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AIRCRAFT LANDING DYNAMIC ANALYSIS
VOLUME 1 EQUATIONS OF MOTION

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This report is Volume 1 of a two volume report. The first volume provides the derivation of the equations of motion and the second volume documents the computer program. The equations of motion are derived for the general case of a flexible aircraft with rigid body motion permitted in six degrees of freedom. The shock strut force equations are derived based on the assumption that a gas-fluid shock strut is used. The hydraulic fluid is assumed to be compressible in the shock strut. The arresting forces are derived for a cable arrestment system. The equations permit landings to be made with unsymmetrical arresting forces. The friction between the cable and arresting hook is accounted for in the derivation of the arresting forces. The equations of motion for the aircraft are developed from Lagrange's equations with the modifications included to use the body axis components of the airframe velocity vector and angular velocity vector as quasi-coordinates.

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FOREWORD

This report was prepared by John W. Lincoln from Structures Division of the Directorate of Flight Systems Engineering, Aeronautical Systems Division, Wright-Patterson Air Force Base Ohio. Its purpose is to provide an analytical method for the accurate and rapid calculation of the loads on an aircraft during landing. The report is written in two volumes. In the first volume the equations of motion are derived and in the second volume the computer program that was developed from these equations of motion is documented.

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INTRODUCTION

The analysis of an aircraft making a landing has interested dynamicists for many years. This interest is derived from the technical challenge of the problem and from the fact that the landing impact is an important loads source for both land and carrier based aircraft. The inherent nonlinear nature of the equations of motion requires that solutions be obtained by numerical integration. The challenge, therefore, is in establishing the equations such that solutions may be obtained with speed and accuracy. Speed is required because of the need to minimize the computer costs derived from the large number of cases that must be examined. Accuracy is essential because it will reduce the need for ground testing to establish the forces and the energy absorbing capability for a given landing gear geometry. Further, and likely more important, an accurate analysis permits one to extrapolate test conditions to design conditions that are difficult or even hazardous to achieve by flight test.

The landing impact analysis in this report is believed to offer both speed and accuracy. Speed is obtained through taking into account the compressibility of the hydraulic fluid in the shock strut. The compressibility is accounted for in the equations of motion through the introduction of an additional first order differential equation. This, in effect, allows the integration to proceed more rapidly than that obtainable if the usual incompressible hydraulic fluid assumption is used.

Accounting for the compressibility of the hydraulic fluid enables the analyst to improve the accuracy of the solution. In many shock struts,

particularly in articulated landing gears, the pressures are large enough to be significantly affected by the hydraulic fluid compressibility. The details of the shock strut equations that account for compressibility are found in Appendix B. Other features in this derivation that improve the accuracy of the solution are six rigid body degrees of freedom, flexible body degrees of freedom for the airframe and representation of the articulated kinematics of the landing gears in combination their flexible degrees of freedom.

The equations of motion are derived with the stroke function of each of the shock struts used as a generalized coordinate. This feature is especially helpful when the gear is articulated. In this case the gear kinematics and gear mass, damping and stiffness terms become functional dependent on the shock strut stroke. The shock strut stroke function and its derivative are obtained directly from the equations of motion instead of deriving them from the axle displacement and velocity functions.

The equations of motion are derived from Lagrange's equations. The rigid body motion of the aircraft is expressed in terms of the body axis components of the aircraft velocity vector function and body axis components of the aircraft angular velocity vector function. These coordinates are quasi-coordinates and the usual formulation of Lagrange's equations must be modified to account for the fact that these coordinates are not generalized. Lagrange's equations must also account for a non-holonomic constraint condition on the vertical motion of a point on the tire footprint in contact with the ground. This constraint condition is expressed by a relationship involving the velocity and angular velocity terms. It is found that the

Lagrangian multiplier in the equations of motion is the vertical ground reaction on the landing gear tire. The modification of Lagrange's equations for quasi-coordinates and for non-holonomic constraints is given in Appendix C.

The aircraft flexibility is represented by orthogonal vibration modes. The influence of this flexibility is included in the calculation of the shock strut forces. The loading on the aircraft from the landing impact may be calculated from the displacement or acceleration defined by the flexible and/or rigid body coordinates.

Finally, the equations are included for the calculation of the forces from a cable type arresting system. It is assumed that the aircraft is equipped with an arresting hook. The equations are derived so that the case where the arresting hook meets the cable off center and the aircraft motion is unsymmetrical is included. The condition under which the hook will or will not slip on the cable is derived and the resulting cable kinematics are incorporated in the equations. The details of this derivation are given in Appendix F.

DERIVATION OF THE EQUATIONS OF MOTION

The equations of motion for the simulation of the landing impact of an aircraft are derived below. The following degrees of freedom are included for the airframe:

Three translational degrees of freedom for the rigid airframe

Three rotational degrees of freedom for the rigid airframe

Flexible body degrees of freedom for the airframe represented by its vibration modes

For each of the landing gears the following degrees of freedom are included:

Rigid body motion of the gear relative to the airframe that is dependent on the shock strut stroke only

Rigid body motion of the gear relative to the airframe that is dependent on castoring of the gear (all gears are assumed to have this degree of freedom)

Flexible body degrees of freedom for the gear represented by its vibration modes

Rotational motion of the gear wheel about the gear axle

The height above the ground of the ground reference point, C_a, in the wheel midplane, will be included as a generalized coordinate in the formulation of the equations of motion. However, a constraint condition on the height of the airframe reference point, N, above the ground provides the means to eliminate this coordinate from the final equations of motion.

The six rigid body motion of the airframe may be described by several different coordinate systems. For the landing impact problem, it is convenient to use the body axis components of the airframe velocity vector function and the body axis components of the airframe angular velocity vector function. In general, motion associated with any one of the six degrees of freedom can have an influence on the the landing gear and the airframe loading.

The airframe flexibility may have a significant effect on the shock strut forces and consequently must be included in the equations of motion. The airframe flexible body equations are also essential if the airframe dynamic response loading is to be computed. This flexibility may be accounted for by several techniques, however, the vibration modes which are orthogonal appear to be the most efficient. For this derivation it is supposed that these normal modes are available. It is further supposed that the structural damping effects can be simulated adequately by linear viscous damping of each mode independently.

The shock strut stroke function is used as a generalized coordinate in the derivation. Consequently, the shock strut force functions as developed in Appendix B are expressed in terms of the stroke and stroking velocity functions. Appendix B gives the neccessary equations for two of the many possible gas and fluid shock strut configurations. For both of these shock struts the pressure in the high pressure side is calculated from the compression of the fluid. At a time t the rate of change of the fluid compression with time is formulated from the shock strut stroking velocity and the volume of fluid passing through the orifice. Consequently, another

integration is added to calculate the compression of the fluid.

This approach may appear to be inefficient when compared with the usual velocity squared damping assumption. However, this additional integration provides stability to the numerical integration process and consequently permits the use of a larger step size. Treating the fluid as compressible also provides a more accurate representation of the shock strut fluid pressure since for some shock struts (particularly in articulated gears) the effect of compressibility is significant.

There are several unit vector systems included in the derivation of the equations of motion. An inertially fixed set of unit vectors is used as a basis from which the motion of the aircraft is defined. At a time t after the aircraft has contacted the ground surface, an airframe fixed set of unit vectors are oriented in space with yaw, pitch and roll Euler angles for the airframe. To provide some simplification in the development of the geometric data and mass terms, each gear has a set of unit vectors that permits roll and pitch of the gear reference axes relative to the airframe. In addition, a unit vector system that is fixed in the castored gear reference system is provided. Since the wheel plane, because of geometrical constraints, may not be conveniently oriented by the gear fixed unit vectors, a separate coordinate system is fixed in the wheel plane. These wheel-plane unit vectors are used in the derivation of the components of the ground force on the tire. This derivation is given in Appendix E.

Kinetic Energy Formulation

The derivation of the equations of motion is accomplished through Lagrange's equations. The use of these equations is believed to offer a significant advantage over the direct application of Newton's second law because of the relative ease in deriving the equations for the articulated gear geometries. At a time t after the aircraft has contacted the ground surface the kinetic energy of the aircraft is divided between the airframe (i.e. the complete aircraft minus the gears) and the gears. The gear kinetic energy is divided between the wheel and the gear structure minus the wheel. The following definitions are needed:

Q is an inertially fixed ground reference point.

N is a reference point in the airframe. The derivation is made with the assumption that this is not the aircraft center of gravity. However, if the center of gravity is used for N then some simplifications are realized.

T_a is the a gear trunnion to airframe attachment point.

A_a is the point that is common to the a gear wheel midplane and the a gear axle centerline.

C_{F_a} is the point that is contained in the ground surface at the center of pressure of the a gear tire.

C_a is a point that is common to the wheel midplane and the line through the

point C_{F_a} that is perpendicular to the ground.

H is the point of attachment of the arresting hook to the airframe. It is supposed that the arresting hook rotates about an axis through the point H and parallel to the lateral axis of the airframe.

LP is the lateral pivot point in the arresting gear shank.

P is the aerodynamic reference point.

TL is a point in the engine(s) thrust line.

V_B is the aircraft volume containing a set of mass points with each mass point labeled by a Cartesian coordinate system fixed in the airframe which for this purpose is assumed to be in a "jig" or undeformed condition. These mass point labels stay the same when the structure is deformed as a result of external loads, however, in this case the mass points move relative to each other. The concept of a jig condition is useful for the definition of certain vectors that permit the rigid body motion of the structure to be separated from the motion of the structure due to deformation.

N_G is the number of gears on the aircraft, a is in [1, N_G] and V_{G_a} is the a gear volume excluding the wheel. The gear mass points are labeled in a manner similar to that used in the airframe. The jig condition of the gear is defined with the shock strut in the fully extended position.

V_{W_a} is the a gear wheel volume.

ρ_B is the simple surface such that if $(x_B^{\,1}, x_B^{\,2}, x_B^{\,3})$ is in V_B, $\rho_B(x_B^{\,1}, x_B^{\,2}, x_B^{\,3})$ is the mass density at the airframe point labeled $(x_B^{\,1}, x_B^{\,2}, x_B^{\,3})$.

ρ_{G_a} is the simple surface such that if $(x_{G_a}^{\,1}, x_{G_a}^{\,2}, x_{G_a}^{\,3})$ is in V_{G_a}, $\rho_{G_a}(x_{G_a}^{\,1}, x_{G_a}^{\,2}, x_{G_a}^{\,3})$ is the mass density at the a gear (without wheel) point labeled $(x_{G_a}^{\,1}, x_{G_a}^{\,2}, x_{G_a}^{\,3})$.

ρ_{W_a} is the simple surface such that if $(x_{W_a}^{\,1}, x_{W_a}^{\,2}, x_{W_a}^{\,3})$ is in V_{W_a}, $\rho_{W_a}(x_{W_a}^{\,1}, x_{W_a}^{\,2}, x_{W_a}^{\,3})$ is the mass density at the a gear wheel point labeled $(x_{W_a}^{\,1}, x_{W_a}^{\,2}, x_{W_a}^{\,3})$.

$\bar{R}_B$ is the vector function such that if $(x_B^{\,1}, x_B^{\,2}, x_B^{\,3})$ is in V_B and $t \geq 0$, $\bar{R}_B(x_B^{\,1}, x_B^{\,2}, x_B^{\,3}, t)$ is the vector at the time t from the ground reference point Q to the airframe point labeled $(x_B^{\,1}, x_B^{\,2}, x_B^{\,3})$.

$\bar{R}_{G_a}$ is the vector function such that if $(x_{G_a}^{\,1}, x_{G_a}^{\,2}, x_{G_a}^{\,3})$ is in V_{G_a} and $t \geq 0$, $\bar{R}_{G_a}(x_{G_a}^{\,1}, x_{G_a}^{\,2}, x_{G_a}^{\,3}, t)$ is the vector at the time t from the point Q to the gear (without wheel) point labeled $(x_{G_a}^{\,1}, x_{G_a}^{\,2}, x_{G_a}^{\,3})$.

$\bar{R}_{W_a}$ is the vector function such that if $(x_{W_a}^{\,1}, x_{W_a}^{\,2}, x_{W_a}^{\,3})$ is in V_{W_a} and $t \geq 0$, $\bar{R}_{W_a}(x_{W_a}^{\,1}, x_{W_a}^{\,2}, x_{W_a}^{\,3}, t)$ is the vector at the time t from the point Q to an the gear wheel point labeled $(x_{W_a}^{\,1}, x_{W_a}^{\,2}, x_{W_a}^{\,3})$.

Therefore, if T is the simple graph such that if $t \geq 0$, $T(t)$ is the kinetic energy at the time t of the collection of mass points, then with the derivative notation described in Appendix A, T may be expressed by

$$T = 1/2 \int_{V_B} (\bar{R}_{B;I_t})^2 \rho_B \, dV_B + \sum_{a=1}^{N_G} 1/2 \int_{V_{G_a}} (\bar{R}_{G_a;I_t})^2 \rho_{G_a} \, dV_{G_a} \tag{1}$$
$$+ \sum_{a=1}^{N_G} 1/2 \int_{V_{W_a}} (\bar{R}_{W_a;I_t})^2 \rho_{W_a} \, dV_{W_a} .$$

The $\bar{R}_B$ vector function must be expressed in terms of constituent vector functions in order to develop the equations of motion. For this purpose the following definitions are needed:

$\bar{r}$ is the vector function such that if $t \geq 0$, $\bar{r}(t)$ is the vector at the time t from the point Q to the jig condition location of the airframe reference point N labeled $(x_{B_N}^1, x_{B_N}^2, x_{B_N}^3)$.

$\bar{L}_B$ is the vector function such that if (x_B^1, x_B^2, x_B^3) is in V_B and $t \geq 0$, $\bar{L}_B(x_B^1, x_B^2, x_B^3, t)$ is the vector at the time t from the jig condition location of the airframe reference point N to the jig condition location of the airframe point labeled (x_B^1, x_B^2, x_B^3).

$\bar{U}_B$ is the vector function such that if (x_B^1, x_B^2, x_B^3) is in V_B and $t \geq 0$, $\bar{U}_B(x_B^1, x_B^2, x_B^3, t)$ is the vector at the time t from the jig condition location of the airframe point labeled (x_B^1, x_B^2, x_B^3) to the actual location of this point.

Therefore, the vector

$$\bar{R}_B(x_B^{\,1}, x_B^{\,2}, x_B^{\,3}, t) = \bar{r}(t) + \bar{L}_B(x_B^{\,1}, x_B^{\,2}, x_B^{\,3}, t) \tag{2}$$

$$+ \bar{U}_B(x_B^{\,1}, x_B^{\,2}, x_B^{\,3}, t)$$

can be used to derive the vector function $\bar{R}_B$.

The vector functions for expansion of $\bar{R}_{G_a}$ are defined as follows:

$\bar{I}_{T_a}$ is the vector function such that if $t \geq 0$, $\bar{I}_{T_a}(t)$ is the vector at the time t from the jig condition location of the airframe reference point N to the jig condition location of the point T_a.

$\bar{G}_a$ is the vector function such that if $(x_{G_a}^{\,1}, x_{G_a}^{\,2}, x_{G_a}^{\,3})$ is in V_{G_a} and $t \geq 0$, $\bar{G}_a(x_{G_a}^{\,1}, x_{G_a}^{\,2}, x_{G_a}^{\,3}, t)$ is the vector at the time t from the jig condition location of the point T_a to the jig condition location of the a gear point $(x_{G_a}^{\,1}, x_{G_a}^{\,2}, x_{G_a}^{\,3})$. The jig condition for the a gear is determined with the shock strut stroke equal to zero (fully extended gear).

s_a is the simple graph such that if $t \geq 0$, $s_a(t)$ is the a gear stroke at the time t.

Ξ_a is the vector function such that if $(x_{G_a}^{\,1}, x_{G_a}^{\,2}, x_{G_a}^{\,3})$ is in V_{G_a} and $t \geq 0$, $\Xi_a(x_{G_a}^{\,1}, x_{G_a}^{\,2}, x_{G_a}^{\,3}, s_a(t), t)$ is that part of the vector at the time t from the jig condition location of the a gear point

$(x_{G_a}^{1}, x_{G_a}^{2}, x_{G_a}^{3})$ to the actual location of this point which is derived from the rigid body motion of the gear as result of the stroking of the shock strut.

θ_{G_a} is the simple graph such that if $t \geq 0$, $\theta_{G_a}(t)$ is the a gear castor angle at the time t. Normally, only the auxiliary gear is permitted to castor. However, for symmetry of the equations of motion, all of the gears in this derivation will be assumed to have this degree of freedom.

$\bar{U}_{BG_a}$ is the vector function such that if $(x_{G_a}^{1}, x_{G_a}^{2}, x_{G_a}^{3})$ is in V_{G_a} and $t \geq 0$, $\bar{U}_{BG_a}(x_{G_a}^{1}, x_{G_a}^{2}, x_{G_a}^{3}, s_a(t), \theta_{G_a}(t), t)$ is that part of the vector at the time t from the jig condition location of the a gear point $(x_{G_a}^{1}, x_{G_a}^{2}, x_{G_a}^{3})$ to the actual location of this point which is derived from the deformation of the airframe. Note that the shock strut stroke and the castor angle dependence appears explicitly in this vector. This provides for a simpler formulation when the deformation vector is expressed in terms of its components.

$\bar{U}_{G_a}$ is the vector function such that if $(x_{G_a}^{1}, x_{G_a}^{2}, x_{G_a}^{3})$ is in V_{G_a} and $t \geq 0$, $\bar{U}_{G_a}(x_{G_a}^{1}, x_{G_a}^{2}, x_{G_a}^{3}, s_a(t), \theta_{G_a}(t), t)$ is that part of the vector at the time t from the jig condition location of the a gear point $(x_{G_a}^{1}, x_{G_a}^{2}, x_{G_a}^{3})$ to the actual location of this point which is derived from the deformation of the a gear.

Therefore, the vector

$$\bar{R}_{G_a}(x_{G_a}{}^1, x_{G_a}{}^2, x_{G_a}{}^3, t) = \bar{r}(t) + \bar{I}_{T_a}(t) \tag{3}$$

$$+ \bar{G}_a(x_{G_a}{}^1, x_{G_A}{}^2, x_{G_a}{}^3, t) + \bar{\Xi}_a(x_{G_a}{}^1, x_{G_a}{}^2, x_{G_a}{}^3, s_a(t), t)$$

$$+ \bar{U}_{BG_a}(x_{G_a}{}^1, x_{G_a}{}^2, x_{G_a}{}^3, s_a(t), \theta_{G_a}(t), t)$$

$$+ \bar{U}_{G_a}(x_{G_a}{}^1, x_{G_a}{}^2, x_{G_a}{}^3, s_a(t), \theta_{G_a}(t), t)$$

can be used to derive the vector function $\bar{R}_{G_a}$.

The final vector function needed for the kinetic energy formulation is $\bar{R}_{W_a}$. The vector functions used to define $\bar{R}_{W_a}$ are defined as follows:

If $(x_{G_{A_a}}{}^1, x_{G_{A_a}}{}^2, x_{G_{A_a}}{}^3)$ is the label of the a gear axle point, A_a, and if $t \geq 0$, then

$$\bar{g}_{A_a}(t) = \bar{G}_a(x_{G_{A_a}}{}^1, x_{G_{A_a}}{}^2, x_{G_{A_a}}{}^3, t),$$

$$\bar{\xi}_{A_a}(s_a(t), t) = \bar{\Xi}_a(x_{G_{A_a}}{}^1, x_{G_{A_a}}{}^2, x_{G_{A_a}}{}^3, s_a(t), t),$$

$$\bar{U}_{BG_{A_a}}(s_a(t), \theta_{G_a}(t), t) = \bar{U}_{BG_a}(x_{G_{A_a}}{}^1, x_{G_{A_a}}{}^2, x_{G_{A_a}}{}^3, s_a(t), \theta_{G_a}(t), t),$$

$$\bar{U}_{G_{A_a}}(s_a(t), \theta_{G_a}(t), t) = \bar{U}_{G_a}(x_{G_{A_a}}{}^1, x_{G_{A_a}}{}^2, x_{G_{A_a}}{}^3, s_a(t), \theta_{G_a}(t), t),$$

at the time t.

$\bar{W}_a$ is the vector function such that if $(x_{W_a}{}^1, x_{W_a}{}^2, x_{W_a}{}^3)$ is in V_{W_a} and $t \geq 0$, $\bar{W}_a(x_{W_a}{}^1, x_{W_a}{}^2, x_{W_a}{}^3, s_a(t), t)$ is the vector at the time t from the axle point A_a to the jig condition location of the point in the a gear labeled $(x_{W_a}{}^1, x_{W_a}{}^2, x_{W_a}{}^3)$.

Therefore, the vector

$$\bar{R}_{W_a}(x_{W_a}^{1}, x_{W_a}^{2}, x_{W_a}^{3}, t) = \bar{r}(t) + \bar{I}_{T_a}(t) + \bar{g}_{A_a}(t) + \bar{\xi}_{A_a}(s_a(t), t) \qquad (4)$$

$$+ \bar{U}_{BG_{A_a}}(s_a(t), \theta_{G_a}(t), t) + \bar{U}_{G_{A_a}}(s_a(t), \theta_{G_a}(t), t)$$

$$+ \bar{W}_a(x_{W_a}^{1}, x_{W_a}^{2}, x_{W_a}^{3}, s_a(t), t)$$

can be used to derive the vector function $\bar{R}_{W_a}$.

To be useful for the kinetic energy equation (equation (1)), the structural deformation vectors appearing in equations (2) through (4) must be expressed in terms of generalized coordinates. This may be done through the use of the vibration modes of the structure. Orthogonality of these modes is assumed in this derivation since this property is typically used in practice.

Suppose that N_{BE} airframe vibration modes are used to define the airframe deformation. Further suppose that b is an integer in $[1, N_{BE}]$ and that $q_B^{\,b}$ is the simple graph such that if $t \geq 0$, $q_B^{\,b}(t)$ is the bth airframe vibration mode displacement at the time t.

Also, suppose that

$\bar{\Phi}_{B_b}$ is the vector function such that if $(x_B^{1}, x_B^{2}, x_B^{3})$ in V_B and $t \geq 0$, $\bar{\Phi}_{B_b}(x_B^{1}, x_B^{2}, x_B^{3}, t)$ is the bth airframe vibration modal vector at the time t for the airframe point labeled $(x_B^{1}, x_B^{2}, x_B^{3})$,

$\bar{\Phi}_{BG_{a_b}}$ is the vector function such that if $(x_{G_a}^{\ 1}, x_{G_a}^{\ 2}, x_{G_a}^{\ 3})$ is in V_{G_a} and $t \geq 0$, $\bar{\Phi}_{BG_{a_b}}(x_{G_a}^{\ 1}, x_{G_a}^{\ 2}, x_{G_a}^{\ 3}, s_a(t), \theta_{G_a}(t), t)$ is the bth airframe vibration modal vector at the time t for the a gear point labeled $(x_{G_a}^{\ 1}, x_{G_a}^{\ 2}, x_{G_a}^{\ 3})$,

and

$\bar{\Phi}_{BG_{A_{a_b}}}$ is the vector function such that if $t \geq 0$, $\bar{\Phi}_{BG_{A_{a_b}}}(s_a(t), \theta_{G_a}(t), t)$ is the bth airframe vibration modal vector at the time t for the a gear axle point A_a.

Now suppose that for the a gear there are N_{GE_a} gear vibration modes used to define its deformation. Suppose that d is an integer in $[1, N_{GE_a}]$ and that $q_{G_a}^{\ d}$ is a simple graph such that if $t \geq 0$, $q_{G_a}^{\ d}(t)$ is the dth a gear vibration mode displacement at the time t. Also suppose that

$\bar{\Phi}_{G_{a_d}}$ is the vector function such that if $(x_{G_a}^{\ 1}, x_{G_a}^{\ 2}, x_{G_a}^{\ 3})$ is in V_{G_a} and $t \geq 0$, $\bar{\Phi}_{G_{a_d}}(x_{G_a}^{\ 1}, x_{G_a}^{\ 2}, x_{G_a}^{\ 3}, s_a(t), \theta_{G_a}(t), t)$ is the dth a gear vibration modal vector at the time t for the a gear point $(x_{G_a}^{\ 1}, x_{G_a}^{\ 2}, x_{G_a}^{\ 3})$,

and

$\bar{\Phi}_{G_{A_{a_d}}}$ is the vector function such that if $t \geq 0$, $\bar{\Phi}_{G_{A_{a_d}}}(s_a(t), \theta_{G_a}(t), t)$ is the dth a gear vibration modal vector at the time t for the a gear axle point A_a.

Therefore, the deformation vectors may be expressed as follows:

$$\bar{U}_B(x_B^{\,1},\ x_B^{\,2},\ x_B^{\,3},\ t) = \bar{\Phi}_{B_b}(x_B^{\,1},\ x_B^{\,2},\ x_B^{\,3},\ t)\ q_B^{\,b}(t),$$

where summation on the index b is implied.

$$\bar{U}_{BG_a}(x_{G_a}^{\,1},\ x_{G_a}^{\,2},\ x_{G_a}^{\,3},\ s_a(t),\ \theta_{G_a}(t),\ t) =$$

$$\bar{\Phi}_{BG_{a_b}}(x_{G_a}^{\,1},\ x_{G_a}^{\,2},\ x_{G_a}^{\,3},\ s_a(t),\ \theta_{G_a}(t),\ t)\ q_B^{\,b}(t).$$

$$\bar{U}_{BG_{A_a}}(s_a(t),\ \theta_{G_a}(t),\ t) = \bar{\Phi}_{BG_{A_{a_b}}}(s_a(t),\ \theta_{G_a}(t),\ t)\ q_B^{\,b}(t).$$

$$\bar{U}_{G_a}(x_{G_a}^{\,1},\ x_{G_a}^{\,2},\ x_{G_a}^{\,3},\ s_a(t),\ \theta_{G_a}(t),\ t) =$$

$$\bar{\Phi}_{G_{a_d}}(x_{G_a}^{\,1},\ x_{G_a}^{\,2},\ x_{G_a}^{\,3},\ s_a(t),\ \theta_{G_a}(t),\ t)\ q_{G_a}^{\,d}(t).$$

$$\bar{U}_{G_{A_a}}(s_a(t),\ \theta_{G_a}(t),\ t) = \bar{\Phi}_{G_{A_{a_d}}}(s_a(t),\ \theta_{G_a}(t),\ t)\ q_{G_a}^{\,d}(t).$$

At this point in the derivation consideration must be given to the order of magnitude of the deformation of the structure. The general case of large deformations, which is described in Reference (1), could be applied. However, this additional complication for most aircraft structures does not appear justified. Therefore, it will be assumed that the deformation is small enough such that terms in the kinetic energy expression involving the $q_B^{\,b}$ functions and the $q_{G_a}^{\,d}$ functions can be eliminated. Also, it will be assumed that the motion of the body axis reference axes is completely defined by the rigid body equations of motion. This assumption is also justified from the assumption of small structural deformations.

Therefore, from equations (2), (3) and (4) the velocity vectors may be

written with the aid of the following definitions:

$\bar{\Omega}_B$ is the vector function such that if $t \geq 0$, $\bar{\Omega}_B(t)$ is the airframe angular velocity vector at the time t.

$\bar{\Omega}_{G_a}$ is the vector function such that if $t \geq 0$, $\bar{\Omega}_{G_a}(t)$ is the a gear castoring angular velocity vector at the time t.

$\bar{\Omega}_{W_a}(t)$ is the vector function such that if $t \geq 0$, $\bar{\Omega}_{W_a}(t)$ is the a gear wheel angular velocity vector at the time t.

Thus, with the assumption of small deformations, the velocities may be written in the following vector format:

$$\bar{R}_{B;I_t}(x_B{}^1, x_B{}^2, x_B{}^3, t) = \bar{r}'(t) + \bar{\Omega}_B(t) \times \bar{L}_B(x_B{}^1, x_B{}^2, x_B{}^3, t) \tag{5}$$
$$+ \bar{\Phi}_{B_b}(x_B{}^1, x_B{}^2, x_B{}^3, t)\, q_B{}^{b\prime}(t).$$

$$\bar{R}_{G_a;I_t}(x_{G_a}{}^1, x_{G_a}{}^2, x_{G_a}{}^3, t) = \bar{r}'(t) + \bar{\Omega}_B(t) \times \bar{I}_{T_a}(t) \tag{6}$$
$$+ (\bar{\Omega}_B(t) + \bar{\Omega}_{G_a}(t)) \times \bar{G}_a(x_{G_a}{}^1, x_{G_a}{}^2, x_{G_a}{}^3, t)$$
$$+ (\bar{\Omega}_B(t) + \bar{\Omega}_{G_a}(t)) \times \bar{\Xi}_a(x_{G_a}{}^1, x_{G_a}{}^2, x_{G_a}{}^3, s_a(t), t)$$
$$+ \bar{\Xi}_{a;s_a}(x_{G_a}{}^1, x_{G_a}{}^2, x_{G_a}{}^3, s_a(t), t)\, s_a{}'(t)$$
$$+ \bar{\Phi}_{BG_{a_b}}(x_{G_a}{}^1, x_{G_a}{}^2, x_{G_a}{}^3, s_a(t), \theta_{G_a}(t), t)\, q_B{}^{b\prime}(t)$$
$$+ \bar{\Phi}_{G_{a_d}}(x_{G_a}{}^1, x_{G_a}{}^2, x_{G_a}{}^3, s_a(t), \theta_{G_a}(t), t)\, q_{G_a}{}^{d\prime}(t).$$

$$\bar{R}_{W_a;I_t}(x_{W_a}^{\ 1}, x_{W_a}^{\ 2}, x_{W_a}^{\ 3}, t) = \bar{r}'(t) + \bar{\Omega}_B(t) \times \bar{I}_{T_a}(t) \tag{7}$$

$$+ (\bar{\Omega}_B(t) + \bar{\Omega}_{G_a}(t)) \times \bar{g}_{A_a}(t) + (\bar{\Omega}_B(t) + \bar{\Omega}_{G_a}(t)) \times \bar{\xi}_{A_a}(s_a(t), t)$$

$$+ \bar{\xi}_{A_{a;s_a}}(s_a(t), t)\, s_a'(t) + \bar{\Phi}_{BG_{A_{a_b}}}(s_a(t), \theta_{G_a}(t), t)\, q_B^{\ b}{}'(t)$$

$$+ \bar{\Phi}_{G_{A_{a_d}}}(s_a(t), \theta_{G_a}(t), t)\, q_{G_a}^{\ d}{}'(t)$$

$$+ (\bar{\Omega}_B(t) + \bar{\Omega}_{G_a}(t) + \bar{\Omega}_{W_a}(t)) \times \bar{W}_a(x_{W_a}^{\ 1}, x_{W_a}^{\ 2}, x_{W_a}^{\ 3}, s_a(t), t)$$

$$+ \bar{W}_{a;s_a}(x_{W_a}^{\ 1}, x_{W_a}^{\ 2}, x_{W_a}^{\ 3}, s_a(t), t)\, s_a'(t).$$

The vector functions $\bar{R}_{B;I_t}$, $\bar{R}_{G_a;I_t}$ and $\bar{R}_{W_a;I_t}$ may be easily derived from equations (5), (6), and (7).

Before $\bar{R}_{B;I_t}$, $\bar{R}_{G_a;I_t}$ and $\bar{R}_{W_a;I_t}$ are substituted into the kinetic energy equation (equation (1)) it is convenient to express the vectors appearing in these functions in terms of their components. To accomplish this, several sets of right handed orthogonal unit vector functions need to be defined.

Suppose each of $\bar{I}_{B_b}$; b = 1, 3 is a unit vector function such that if $t \geq 0$, $\bar{I}_{B_1}(t)$ is the vector parallel to the airframe longitudinal axis and directed forward, $\bar{I}_{B_2}(t)$ is parallel to the airframe lateral axis and directed to the right relative to the pilot and $\bar{I}_{B_3}(t)$ is parallel to the airframe vertical axis and directed down relative to the pilot at the time t.

To orient the airframe in space relative to an inertial reference frame defined by the unit vectors $\bar{I}_b$; b = 1, 3, a set of Euler angles is used. For this purpose suppose that each of ψ, θ and ϕ is a simple graph such

that if $t \geq 0$, $\psi(t)$, $\theta(t)$ and $\phi(t)$ are at the time t the yaw, pitch and roll Euler angles of the airframe. These Euler angles are used in the transformation

$$\bar{i}_{B_c}(t) = \gamma_{B_c}{}^{b}(t)\ \bar{I}_b,$$

where the components of $\gamma_{B_c}{}^{b}(t)$ are derived in Appendix G.

From Appendix G it is seen that $\bar{I}_1$ and $\bar{I}_2$ are in the ground plane and $\bar{I}_3$ is perpendicular to the ground.

Also, suppose that each of $\bar{i}_{G_{a_b}}$; b = 1, 3 is a unit vector function and each of η_{G_a} and ζ_{G_a} is a number such that η_{G_a} is the pitch Euler angle and ζ_{G_a} is the roll Euler angle for the a gear relative to the airframe such that if $t \geq 0$, each of $\bar{i}_{G_{a_b}}(t)$; b = 1, 3 are fixed relative to the a gear at the time t and are oriented relative to the airframe fixed unit vectors through the transformation

$$\bar{i}_{G_{a_c}}(t) = \gamma_{G_{a_c}}{}^{b}\ \bar{i}_{B_b}(t),$$

where the components of $\gamma_{G_{a_c}}{}^{b}$ are derived in Appendix G in terms of the Euler angles η_{G_a} and ζ_{G_a}.

Suppose that each of $\bar{j}_{G_{a_b}}$; b = 1, 3 is a unit vector function such that if $t \geq 0$, each of $\bar{j}_{G_{a_b}}(t)$; b = 1, 3 is a vector fixed in the a gear at the

time t. The castor angle, θ_{G_a} is used to orient these vectors with respect to $\bar{i}_{G_{a_b}}(t)$; b = 1, 3 vectors through the transformation

$$\bar{j}_{G_{a_b}}(t) = \hat{\alpha}_{G_{a_g}}{}^{e}(t)\ \bar{i}_{G_{a_c}}(t)\ \delta_{be}\ \delta^{cg},$$

where $\hat{\alpha}_{G_{a_g}}{}^{e}(t)$ is defined in Appendix G.

It is also useful to define the transformation from the airframe fixed unit vectors $\bar{i}_{B_b}(t)$; b = 1, 3 to the a gear fixed unit vectors $\bar{j}_{G_{a_b}}(t)$; b = 1, 3. This may be done through a combination of the transformations defined above as follows:

$$\bar{j}_{G_{a_b}}(t) = \hat{\alpha}_{G_{a_g}}{}^{e}(t)\ \hat{\gamma}_{G_{a_c}}{}^{d}(t)\ \bar{i}_{B_d}(t)\ \delta_{bc}\ \delta^{eg}$$

$$= \alpha_{G_{a_b}}{}^{d}(t)\ \bar{i}_{B_d}(t).$$

The components of $\alpha_{G_{a_b}}{}^{d}$ are given in Appendix G.

Suppose that each of $\bar{k}_{G_{a_b}}$; b = 1, 3 is a unit vector function such that if $t \geq 0$, each of $\bar{k}_{G_{a_b}}(t)$; b = 1, 3 is fixed in the a gear wheel at the time t and $\bar{k}_{G_{a_2}}(t)$ is a unit vector parallel to the a gear wheel axle. Therefore, the vector $\bar{k}_{G_{a_2}}(t)$ may be expressed in terms of the vectors $\bar{j}_{G_{a_b}}(t)$; b = 1, 3 by the relation

$$\bar{k}_{G_{a_2}}(t) = \beta_{G_{a_2}}{}^{b}(t)\ \bar{j}_{G_{a_b}}(t),$$

where from the gear kinematics there exists a set of functions

$B_{G_{a_2}}{}^{b}$; b = 1, 3 such that

$$\beta_{G_{a_2}}{}^{b}(t) = B_{G_{a_2}}{}^{b}(s_a(t));\ b = 1,\ 3.$$

Now suppose that each of $\bar{I}_{W_{a_b}}$; b = 1, 3 is a unit vector function such that if $t \geq 0$, $\bar{I}_{W_{a_3}}(t)$ is $\bar{I}_3$, $\bar{I}_{W_{a_1}}(t)$ is orthogonal to $\bar{k}_{G_{a_2}}(t)$ and $\bar{I}_{W_{a_3}}(t)$, and $\bar{I}_{W_{a_2}}(t)$ is orthogonal to $\bar{I}_{W_{a_1}}(t)$ and $\bar{I}_{W_{a_3}}(t)$ at the time t.

The transformation from $\bar{I}_b$; b = 1, 3 to $\bar{I}_{W_{a_b}}(t)$; b =1, 3 is defined in Appendix E and is given by

$$\bar{I}_{W_{a_b}}(t) = \gamma_{W_{a_g}}{}^{e}(t)\ \bar{I}_c\ \delta_{be}\ \delta^{cg}.$$

With the aid of the unit vector functions defined above the linear and angular velocity functions are defined by

$$\bar{r}' = V_B{}^{b}\ \bar{I}_{B_b},$$

$$\bar{\Omega}_B = \Omega_B{}^{b}\ \bar{I}_{B_b},$$

$$\bar{\Omega}_{G_a} = \Omega_{G_a}\ \bar{j}_{G_{a_3}},$$

$$\bar{\Omega}_{W_a} = \Omega_{W_a}\ \bar{k}_{G_{a_2}}.$$

The vector functions in equations (5), (6) and (7) may be defined by

$$\bar{L}_B = L_B{}^b\, \bar{i}_{B_b},$$

$$\bar{l}_{T_a} = l_{T_a}{}^b\, \bar{i}_{B_b},$$

$$\bar{G}_a = G_a{}^b\, \bar{j}_{G_{a_b}},$$

$$\bar{g}_{A_a} = g_{A_a}{}^b\, \bar{j}_{G_{a_b}},$$

$$\bar{\Xi}_a = \Xi_a{}^b\, \bar{j}_{G_{a_b}},$$

$$\bar{\xi}_{A_a} = \xi_{A_a}{}^b\, \bar{j}_{G_{a_b}},$$

$$\bar{W}_a = W_a{}^b\, \bar{k}_{G_{a_b}}.$$

Finally, the modal vector functions for the deformation of the airframe and the gears are defined by

$$\bar{\Phi}_{B_b} = \Phi_{B_b}{}^c\, \bar{i}_{B_c},$$

$$\bar{\Phi}_{BG_{a_b}} = \Phi_{BG_{a_b}}{}^c\, \bar{i}_{G_{a_c}},$$

$$\bar{\Phi}_{BG_{A_{a_b}}} = \Phi_{BG_{A_{a_b}}}{}^c\, \bar{i}_{G_{a_c}},$$

$$\bar{\Phi}_{G_{a_b}} = \Phi_{G_{a_b}}{}^c\, \bar{j}_{G_{a_c}},$$

$$\bar{\Phi}_{G_{A_{a_b}}} = \Phi_{G_{A_{a_b}}}{}^c\, \bar{j}_{G_{a_c}}.$$

Therefore, with these definitions the kinetic energy function T may be expressed in the following form:

$$
\begin{aligned}
T = {} & 1/2\ (M^{V_B V_B}_{b\ c})\, V_B^{\,b}\, V_B^{\,c} + 1/2\ (M^{\Omega_B \Omega_B}_{b\ c})[s_a,\ \theta_{G_a};\ a = 1,\ N_G]\, \Omega_B^{\,b}\, \Omega_B^{\,c} \qquad (8) \\
& + 1/2 \sum_{a=1}^{N_G} (M^{\Omega_{G_a} \Omega_{G_a}})[s_a]\, (\Omega_{G_a})^2 + 1/2 \sum_{a=1}^{N_G} (M^{\Omega_{W_a} \Omega_{W_a}})\, (\Omega_{W_a})^2 \\
& + 1/2 \sum_{a=1}^{N_G} (M^{s_a s_a})[s_a]\, (s_a')^2 + 1/2\ (M^{q_B q_B}_{b\ c})[s_a,\ \theta_{G_a};\ a = 1,\ N_G]\, q_B^{\,b'}\, q_B^{\,c'} \\
& + 1/2 \sum_{a=1}^{N_G} (M^{q_{G_a} q_{G_a}}_{b\ c})[s_a,\ \theta_{G_a}]\, q_{G_a}^{\,b'}\, q_{G_a}^{\,c'} \\
& + (M^{V_B \Omega_B}_{b\ c})[s_a,\ \theta_{G_a};\ a = 1,\ N_G]\, V_B^{\,b}\, \Omega_B^{\,c} \\
& + \sum_{a=1}^{N_G} (M^{V_B \Omega_{G_a}}_{b})[s_a,\ \theta_{G_a}]\, V_B^{\,b}\, \Omega_{G_a} + \sum_{a=1}^{N_G} (M^{V_B s_a}_{b})[s_a,\ \theta_{G_a}]\, V_B^{\,b}\, s_a' \\
& + (M^{V_B q_B}_{b\ c})[s_a,\ \theta_{G_a};\ a = 1,\ N_G]\, V_B^{\,b}\, q_B^{\,c'} + \sum_{a=1}^{N_G} (M^{V_B q_{G_a}}_{b\ c})[s_a,\ \theta_{G_a}]\, V_B^{\,b}\, q_{G_a}^{\,c'} \\
& + \sum_{a=1}^{N_G} (M^{\Omega_B \Omega_{G_a}}_{b})[s_a,\ \theta_{G_a}]\, \Omega_B^{\,b}\, \Omega_{G_a} + \sum_{a=1}^{N_G} (M^{\Omega_B \Omega_{W_a}}_{b})[s_a,\ \theta_{G_a}]\, \Omega_B^{\,b}\, \Omega_{W_a} \\
& + \sum_{a=1}^{N_G} (M^{\Omega_B s_a}_{b})[s_a,\ \theta_{G_a}]\, \Omega_B^{\,b}\, s_a' + (M^{\Omega_B q_B}_{b\ c})[s_a,\ \theta_{G_a};\ a = 1,\ N_G]\, \Omega_B^{\,b}\, q_B^{\,c'} \\
& + \sum_{a=1}^{N_G} (M^{\Omega_B q_{G_a}}_{b\ c})[s_a,\ \theta_{G_a}]\, \Omega_B^{\,b}\, q_{G_a}^{\,c'} \\
& + \sum_{a=1}^{N_G} (M^{\Omega_{G_a} \Omega_{W_a}})[s_a]\, \Omega_{G_a}\, \Omega_{W_a} + \sum_{a=1}^{N_G} (M^{\Omega_{G_a} s_a})[s_a]\, \Omega_{G_a}\, s_a'
\end{aligned}
$$

$$+ \sum_{a=1}^{N_G} (M^{\Omega_{G_a} q_B}{}_b)[s_a, \theta_{G_a}] \Omega_{G_a} q_B{}^{b\prime} + \sum_{a=1}^{N_G} (M^{\Omega_{G_a} q_{G_a}}{}_b)[s_a, \theta_{G_a}] \Omega_{G_a} q_{G_a}{}^{b\prime}$$

$$+ \sum_{a=1}^{N_G} (M^{s_a q_B}{}_b)[s_a, \theta_{G_a}] s_a' q_B{}^{b\prime} + \sum_{a=1}^{N_G} (M^{s_a q_{G_a}}{}_b)[s_a, \theta_{G_a}] s_a' q_{G_a}{}^{b\prime}$$

$$+ \sum_{a=1}^{N_G} (M^{q_B q_{G_a}}{}_{b\,c})[s_a, \theta_{G_a}] q_B{}^{b\prime} q_{G_a}{}^{c\prime},$$

where the mass terms in the expression above are defined in Appendix D and the bracket product notation is defined in Appendix A.

Suppose that K is a simple surface such that the kinetic energy function T given by equation (8) may be written in the form

$$T = K[V_B{}^1, V_B{}^2, V_B{}^3, \Omega_B{}^1, \Omega_B{}^2, \Omega_B{}^3, \Omega_{G_1}, \ldots, \Omega_{G_{NG}}, \Omega_{W_1}, \ldots, \Omega_{W_{NG}}, s_1', \ldots, s_{N_G}', q_B{}^{1\prime}, \ldots, q_B{}^{N_{BE}\prime}, q_{G_1}{}^{1\prime}, \ldots, q_{G_{NG}}{}^{N_{GE_{NG}}\prime}, \theta_{G_1}, \ldots, \theta_{G_{NG}}, s_1, \ldots, s_{NG}]. \tag{9}$$

It is seen that the function T formulated in equation (9) is expressed partially in terms of coordinates (i.e. functions) referred to in Reference (2) as quasi-coordinates. Namely, the body axis components of the airframe velocity vector function and the body axis components of the airframe angular velocity vector function are quasi-coordinates. Appendix C describes the modifications to Lagrange's equations when quasi-coordinates are used.

Suppose that ζ^a; a = 1, 3 are the components of $\bar{r}$ (the vector from the

point Q to the jig condition location of the point N) in the inertial reference frame defined by $\bar{I}_b$; b = 1, 3. It is useful (e.g. in Appendix F) to use the symbols d, s and v as alternate designations for these components.

Therefore, if Ψ, θ and ϕ are the yaw, pitch and roll Euler angle functions which are discussed in Appendix G, then consistent with the notation of Appendix C, the following generalized coordinates are used for the airframe rigid body motion:

$$q^1 = \zeta^1 = d. \qquad q^4 = \theta.$$
$$q^2 = \zeta^2 = s. \qquad q^5 = \phi.$$
$$q^3 = \zeta^3 = v. \qquad q^6 = \Psi.$$

Also, the following relations are consistent with Appendix C notation:

$$v^1 = v_B{}^1. \qquad v^4 = \Omega_B{}^1.$$
$$v^2 = v_B{}^2. \qquad v^5 = \Omega_B{}^2.$$
$$v^3 = v_B{}^3. \qquad v^6 = \Omega_B{}^3.$$

The transformation from the quasi-coordinates to the generalized coordinates is

$$q^{a\prime} = \beta^a_b \, v^b,$$

where from Appendix G it is found that

$$\beta^1_1 = \cos[\theta] \cos[\Psi], \qquad \beta^1_2 = -\sin[\Psi] \cos[\phi] + \sin[\phi] \sin[\theta] \cos[\Psi], \qquad \beta^1_3 = \sin[\Psi] \sin[\phi] + \cos[\phi] \sin[\theta] \cos[\Psi],$$

$$\beta^2_1 = \cos[\theta] \sin[\Psi], \qquad \beta^2_2 = \cos[\phi] \cos[\Psi] + \sin[\phi] \sin[\theta] \sin[\Psi], \qquad \beta^2_3 = -\sin[\phi] \cos[\Psi] + \cos[\phi] \sin[\theta] \sin[\Psi],$$

β^3_1 = - sin[θ], β^3_2 = sin[ϕ] cos[θ], β^3_3 = cos[ϕ] cos[θ],

β^4_4 = 0, β^4_5 = cos[ϕ], β^4_6 = - sin[ϕ],

β^5_4 = 1, β^5_5 = tan[θ] sin[ϕ], β^5_6 = tan[θ] cos[ϕ],

β^6_4 = 0, β^6_5 = sin[ϕ]/cos[θ], β^6_6 = cos[ϕ]/cos[θ],

and all other β^a_b = 0.

The elements of the inverse transformation

$v^a = \alpha^a_b \, q^b$, are

α^1_1 = cos[θ] cos[ψ], α^1_2 = cos[θ] sin [ψ], α^1_3 = - sin[θ],

α^2_1 = - sin[ψ] cos[ϕ] + sin[ϕ] sin[θ] cos[ψ], α^2_2 = cos[ϕ] cos[ψ] + sin[ϕ] sin[θ] sin[ψ], α^2_3 = sin[ϕ] cos[θ],

α^3_1 = sin[ψ] sin[ϕ] + cos[ϕ] sin[θ] cos[ψ], α^3_2 = - sin[ϕ] cos[ψ] + cos[ϕ] sin[θ] sin[ψ], α^3_3 = cos[ϕ] cos[θ],

α^4_4 = 0, α^4_5 = 1, α^4_6 = - sin[θ],

α^5_4 = cos[ϕ], α^5_5 = 0, α^5_6 = cos[θ] sin[ϕ],

α^6_4 = - sin[ϕ], α^6_5 = 0, α^6_6 = cos[θ] cos[ϕ],

and all other α^a_b = 0.

For a in [1, 6] and b in [1, 6], the Ω_b^a functions which are defined in terms of α_b^a and β_b^a in equation (C-11) of Appendix C are

$$\Omega_1^1 = 0, \qquad \Omega_2^1 = -\Omega_B{}^3, \qquad \Omega_3^1 = \Omega_B{}^2,$$

$$\Omega_4^1 = 0, \qquad \Omega_5^1 = -\ V_B{}^3, \qquad \Omega_6^1 = V_B{}^2,$$

$$\Omega_1^2 = \Omega_B{}^3, \qquad \Omega_2^2 = 0, \qquad \Omega_3^2 = -\Omega_B{}^1,$$

$$\Omega_4^2 = V_B{}^3, \qquad \Omega_5^2 = 0, \qquad \Omega_6^2 = -\ V_B{}^1,$$

$$\Omega_1^3 = -\Omega_B{}^2, \qquad \Omega_2^3 = \Omega_B{}^1, \qquad \Omega_3^3 = 0,$$

$$\Omega_4^3 = -\ V_B{}^2, \qquad \Omega_5^3 = V_B{}^1, \qquad \Omega_6^3 = 0,$$

$$\Omega_4^4 = 0, \qquad \Omega_5^4 = -\Omega_B{}^3, \qquad \Omega_6^4 = \Omega_B{}^2,$$

$$\Omega_4^5 = \Omega_B{}^3, \qquad \Omega_5^5 = 0, \qquad \Omega_6^5 = \Omega_B{}^1,$$

$$\Omega_4^6 = -\Omega_B{}^2, \qquad \Omega_5^6 = \Omega_B{}^1, \qquad \Omega_6^6 = 0,$$

and all other $\Omega_b^a = 0$.

The Forces Acting on the System

The quasi-coordinate force terms may be defined through equation (C-18) of Appendix C. Included in these terms are the body gravitational force, the arresting gear force, the ground force (parallel to the ground plane), the aerodynamic force and the thrust force. There are also some generalized coordinate force terms that must be included in the equations of motion. These include the shock strut forces, the airframe modal elastic and damping forces, the gear modal elastic and damping forces and the gear stiffness and damping forces associated with castoring. The notation on the left side of equation (C-18) will be used for both quasi and generalized coordinates.

Consistent with the definitions given in Appendix C and with g equal to the gravitational acceleration the virtual work done by the gravity force is

$$\delta W_W(t) = \int_{V_B} (\delta \bar{R}_B (I_{x_B}{}^1, I_{x_B}{}^2, I_{x_B}{}^3, t) \cdot \bar{I}_3) \rho_B \, g \, dV_B$$

$$+ \sum_{a=1}^{N_G} \int_{V_{G_a}} (\delta \bar{R}_{G_a} (I_{x_{G_a}}{}^1, I_{x_{G_a}}{}^2, I_{x_{G_a}}{}^3, t) \cdot \bar{I}_3) \rho_{G_a} \, g \, dV_{G_a}$$

$$+ \sum_{a=1}^{N_G} \int_{V_{W_a}} (\delta \bar{R}_{W_a} (I_{x_{W_a}}{}^1, I_{x_{W_a}}{}^2, I_{x_{W_a}}{}^3, t) \cdot \bar{I}_3) \rho_{W_a} \, g \, dV_{W_a} .$$

Therefore, based on Appendix C, the vector functions $\delta \bar{R}_B$, $\delta \bar{R}_{G_a}$ and $\delta \bar{R}_{W_a}$ may be replaced by $\bar{R}_{B;I_t}$, $\bar{R}_{G_a;I_t}$ and $\bar{R}_{W_a;I_t}$ respectively.

Thus,

$$\delta W_W(t) = \int_{V_B} (\bar{R}_{B;I_t} (I_{x_B}{}^1, I_{x_B}{}^2, I_{x_B}{}^3, t) \cdot \bar{I}_3)\, \rho_B\, g\, dV_B \tag{10}$$

$$+ \sum_{a=1}^{N_G} \int_{V_{G_a}} (\bar{R}_{G_a;I_t} (I_{x_{G_a}}{}^1, I_{x_{G_a}}{}^2, I_{x_{G_a}}{}^3, t) \cdot \bar{I}_3)\, \rho_{G_a}\, g\, dV_{G_a}$$

$$+ \sum_{a=1}^{N_G} \int_{V_{W_a}} (\bar{R}_{W_a;I_t} (I_{x_{W_a}}{}^1, I_{x_{W_a}}{}^2, I_{x_{W_a}}{}^3, t) \cdot \bar{I}_3)\, \rho_{W_a}\, g\, dV_{W_a}.$$

If equations (5), (6) and (7) are substituted into equation (10) and if the mass terms defined in Appendix D are utilized, then the following functions are found to be the coefficients of the coordinate velocities:

$$S_W{}^{V_B}{}_b(t) = g\, (M^{V_B\, V_B}{}_{b\,c})\, \gamma_{B_d}{}^3(t)\, \delta^{cd} \qquad (b = 1,\ 3).$$

$$S_W{}^{\Omega_B}{}_b(t) = g\, (M^{V_B\, \Omega_B}{}_{c\,b})(s_a(t),\ \theta_{G_a}(t);\ a = 1,\ N_G)\, \gamma_{B_d}{}^3(t)\, \delta^{cd} \qquad (b = 1,\ 3).$$

$$S_W{}^{\Omega_{G_a}}(t) = g\, (M^{V_B\, \Omega_{G_a}}{}_c)(s_a(t),\ \theta_{G_a}(t))\, \gamma_{B_d}{}^3(t)\, \delta^{cd}.$$

$$S_W{}^{s_a'}(t) = g\, (M^{V_B\, s_a}{}_c)(s_a(t),\ \theta_{G_a}(t))\, \gamma_{B_d}{}^3(t)\, \delta^{cd}.$$

$$S_W{}^{q_B'}{}_b(t) = g\, (M^{V_B\, q_B}{}_{c\,b})(s_a(t),\ \theta_{G_a}(t);\ a = 1,\ N_G)\, \gamma_{B_d}{}^3(t)\, \delta^{cd} \qquad (b = 1,\ N_{BE}).$$

$$S_W{}^{q_{G_a}'}{}_b(t) = g\, (M^{V_B\, q_{G_a}}{}_{c\,b})(s_a(t),\ \theta_{G_a}(t))\, \gamma_{B_d}{}^3(t)\, \delta^{cd} \qquad (b = 1,\ N_{GE_a}).$$

These functions are the contributions from the body gravity force to

the total quasi-coordinate force and will be used in the equations of motion (equation C-16). For example, the simple graph $S_{W_b}^{V_B}$ will be included on the right side of the $V_B^{\,b}$ equation of motion.

Suppose that $\bar{F}_{HP}$ is the vector function such that if $t \geq 0$, $\bar{F}_{HP}(t)$ is the force at the time t acting on the point HP from the arresting hook. The point HP is defined in Appendix F. Also, at the time t, $\bar{F}_H(t)$ is the force on the lateral pivot point, LP, from the lower arresting hook segment and is directed along a line from LP to HP. $\bar{F}_H(t)$ is equal to $-\bar{F}_{HP}(t)$.

Further, suppose that $\bar{r}_{LP}(t)$ is the vector function such that if $t \geq 0$, $\bar{r}_{LP}(t)$ is the vector from the ground reference point Q to the point labeled LP at the time t.

The virtual work done by the arresting force is therefore

$$\delta W_H(t) = \bar{F}_H(t) \cdot \delta \bar{r}_{LP}(t).$$

Now suppose that

$\bar{l}_H$ is the vector function such that if $t \geq 0$, $\bar{l}_H(t)$ is the vector from the jig condition location of the airframe reference point N to the jig condition location of the point labeled H, which is the point of attachment of the arresting hook to the airframe, at the time t,

$\bar{l}_{LP}$ is the vector function such that if $t \geq 0$, $\bar{l}_{LP}(t)$ is the vector from the jig condition location of the point H to the point LP at the time t,

$\bar{\Phi}_{B_{H_b}}$ is the vector function such that if $t \geq 0$, $\bar{\Phi}_{B_{H_b}}(t)$ is the bth airframe modal vector at the time t for the point labeled H,

$\bar{\Psi}_{B_{H_b}}$ is the vector function such that if $t \geq 0$, $\bar{\Psi}_{B_{H_b}}(t)$ is the bth airframe modal slope vector at the time t for the point labeled H and

$\bar{i}_{H_1}(t)$, $\bar{i}_{H_2}(t)$ and $\bar{i}_{H_3}(t)$ are the unit vector functions such that if $t \geq 0$,

$\bar{i}_{H_1}(t)$ has the direction of $-\bar{l}_{LP}(t)$, $\bar{i}_{H_2}(t)$ is $\bar{i}_{B_2}(t)$, and

$\bar{i}_{H_3}(t)$ is orthogonal to $\bar{i}_{H_1}(t)$ and $\bar{i}_{H_2}(t)$ and is directed down with respect to the pilot. See Appendix F for derivation of these vectors.

Therefore, the vector $\bar{r}_{LP}(t)$ may be written as

$$\bar{r}_{LP}(t) = \bar{r}(t) + \bar{l}_H(t) + \bar{l}_{LP}(t) + \bar{\Phi}_{B_{H_b}}(t)\, q_B{}^b(t)$$
$$+ (\bar{\Psi}_{B_{H_b}}(t) \cdot \bar{i}_{H_3}(t))\, \bar{i}_{H_3}(t) \times \bar{l}_{LP}(t)\, q_B{}^b(t).$$

Based on the assumption of small airframe deformations the vector function $\bar{r}_{LP}$ may be differentiated to obtain

$$\bar{r}_{LP}'(t) = \bar{r}'(t) + \bar{\Omega}_B(t) \times \bar{l}_H(t) + (\bar{\Omega}_B(t) \cdot \bar{i}_{H_3}(t))\, \bar{i}_{H_3}(t) \times \bar{l}_{LP}(t) \quad (11)$$
$$+ \bar{\Phi}_{B_{H_b}}(t)\, q_B{}^{b\prime}(t) + (\bar{\Psi}_{B_{H_b}}(t) \cdot \bar{i}_{H_3}(t))\, \bar{i}_{H_3}(t) \times \bar{l}_{LP}(t)\, q_B{}^{b\prime}(t).$$

The vector $\bar{r}_{LP}'(t)$ may be used as a replacement for $\delta\bar{r}_H(t)$ in the virtual work equation for the arresting gear force. The following

definitions may be used to expand the vectors in equation (11):

$F_H^{\ b}$ is the simple graph such that if $t \geq 0$,

$\bar{F}_H(t) = F_H^{\ b}(t)\ \bar{I}_b$ at the time t.

Each of $\alpha_{H_3}^{\ c}$ and $\gamma_{H_c}^{\ d}$; c = 1, 3; d = 1, 3 is the simple graph derived in Appendix F such that if $t \geq 0$,

$\bar{I}_{H_3}(t) = \alpha_{H_3}^{\ c}(t)\ \gamma_{H_c}^{\ d}(t)\ \bar{I}_d$ at the time t.

a_1 and b_1 are the simple graphs derived in Appendix F such that if $t \geq 0$,

$\bar{I}_{LP}(t) = a_1(t)\ \bar{i}_{B_1}(t) + b_1(t)\ \bar{i}_{B_3}(t)$ at the time t.

$\bar{r}'(t) = v_B^{\ c}(t)\ \bar{i}_{B_c}(t).$

$\bar{I}_H(t) = l_H^{\ d}\ \bar{i}_{B_d}(t).$

$\bar{\Phi}_{B_{H_b}}(t) = \Phi_{B_{H_b}}^{\ \ c}\ \bar{i}_{B_c}(t).$

$\bar{\Psi}_{B_{H_b}}(t) = \Psi_{B_{H_b}}^{\ \ c}\ \bar{i}_{B_c}(t).$

It follows then that after the indicated vector operations on equation (11) are performed the following coefficients of the coordinate velocities may be interpreted as the quasi-coordinate and generalized coordinate force contributions from the arresting force:

$$S_H^{V_B}{}_b(t) = \gamma_{B_b}{}^d(t)\, F_H{}^c(t)\, \delta_{cd} \qquad (b = 1, 3).$$

$$S_H^{\Omega_B}{}_c(t) = \gamma_{B_b}{}^h(t)\, F_H{}^g(t)\, e_{cdf}\, 1_H{}^d\, \delta^{fb}\, \delta_{gh}$$

$$+ (\gamma_{B_c}{}^g(t)\, \alpha_{H_3}{}^h(t)\, \gamma_{H_h}{}^p(t)\, \delta_{pg})\, e_{sdr}\, F_H{}^b(t)$$

$$\times\, \alpha_{H_3}{}^f(t)\, \gamma_{H_f}{}^d(t)\, (a_1(t)\, \gamma_{B_1}{}^r(t) + b_1(t)\, \gamma_{B_3}{}^r(t))\, \delta_b^s \qquad (c = 1, 3).$$

$$S_H^{q_B'}{}_b(t) = \gamma_{B_c}{}^f(t)\, \Phi_{B_{H_b}}{}^c\, F_H{}^g(t)\, \delta_{fg}$$

$$+ (\Psi_{B_{H_b}}{}^c\, \gamma_{B_c}{}^g(t)\, \alpha_{H_3}{}^h(t)\, \gamma_{H_h}{}^p(t)\, \delta_{pg})\, e_{sdr}\, F_H{}^q(t)$$

$$\times\, \alpha_{H_3}{}^f(t)\, \gamma_{H_f}{}^d(t)\, (a_1(t)\, \gamma_{B_1}{}^r(t) + b_1(t)\, \gamma_{B_3}{}^r(t))\, \delta_q^s \qquad (b = 1, N_{BE}).$$

Now suppose that $\bar{F}_{G_a}$ is the vector function such that if $t \geq 0$, $\bar{F}_{G_a}(t)$ is the force on the a gear wheel from the ground at the time t.

Also, suppose that $\bar{r}_{C_{F_a}}$ is the vector function such that if $t \geq 0$, $\bar{r}_{C_{F_a}}(t)$ is the vector from the ground reference point Q to the point C_{F_a} (the a wheel tire footprint center of pressure).

Therefore, the virtual work of the force from the ground on the tires is

$$\delta W_G(t) = \sum_{a=1}^{N_G} \bar{F}_{G_a}(t) \cdot \delta \bar{r}_{C_{F_a}}(t).$$

It is noted that this expression is the virtual work done by the ground forces parallel to the ground plane. These ground forces are derived in Appendix E. The generalized forces from the ground which are normal to the ground plane will be derived from the potential energy stored in the tires.

To make the virtual work expression above more usable the vector $\bar{r}_{C_{F_a}}(t)$ must be expressed in terms of the system coordinates. This may be accomplished as follows:

Suppose $\bar{w}_{C_a}$ is the vector function such that if $t \geq 0$, $\bar{w}_{C_a}(t)$ is the vector at the time t from the point A_a (the a gear axle reference point) to the point C_a (a point that is common to the a gear wheel centerplane and the vertical ground force vector).

Further, suppose that $\bar{v}_{W_a}$ is the vector function such that if $t \geq 0$, $\bar{v}_{W_a}(t)$ is the vector from the point C_a to the point C_{F_a} at the time t. In addition, suppose that v_{W_a} is the simple graph such that if $t \geq 0$ then $\bar{v}_{W_a}(t) = v_{W_a}(t)\, \bar{I}_3$.

The vector $\bar{r}_{C_{F_a}}(t)$ may be expressed as follows:

$$\bar{r}_{C_{F_a}}(t) = \bar{r}(t) + \bar{I}_{T_a}(t) + \bar{\Phi}_{BG_{A_{a_b}}}(s_a(t), \theta_{G_a}(t), t)\, q_B{}^b(t) \tag{12}$$

$$+ \bar{g}_{A_a}(t) + \bar{\xi}_{A_a}(s_a(t), t) + \bar{\Phi}_{G_{A_{a_b}}}(s_a(t), \theta_{G_a}(t), t)\, q_{G_a}{}^b(t) + \bar{w}_{C_a}(t) + \bar{v}_{W_a}(t).$$

If the vector function $\bar{r}_{C_{F_a}}$ is differentiated and it is assumed that

the magnitude of the vector $\bar{w}_{C_a}(t)$ is independent of time then

$$\bar{r}_{C_{F_a}}'(t) = \bar{V}_B(t) + \bar{\Omega}_B(t) \times \bar{I}_{T_a}(t) \tag{13}$$

$$+ \bar{\Omega}_B(t) \times (\bar{g}_{A_a}(t) + \bar{\xi}_{A_a}(s_a(t), t) + \bar{w}_{C_a}(t) + \bar{v}_{W_a}(t))$$

$$+ \bar{\Omega}_{a}(t) \times (\bar{g}_{A_a}(t) + \bar{\xi}_{A_a}(s_a(t), t) + \bar{w}_{C_a}(t) + \bar{v}_{W_a}(t))$$

$$+ \bar{\Omega}_{W_a}(t) \times (\bar{w}_{C_a}(t) + \bar{v}_{W_a}(t))$$

$$+ \bar{\Phi}_{BG_{A_{a_b}}}(s_a(t), \theta_{G_a}(t), t)\, q_B{}^{b}{}'(t) + \bar{\xi}_{A_a;s_a}(t)\, s_a'(t)$$

$$+ \bar{\Phi}_{G_{A_{a_b}}}(s_a(t), \theta_{G_a}(t), t)\, q_{G_a}{}^{b}{}'(t) + \bar{I}_3\, v_{W_a}'(t).$$

Since at the time t the vector $\bar{w}_{C_a}(t)$ is in the wheel midplane from the point A_a to the point C_a, it follows that $\bar{w}_{C_a}$ may be expressed by a unit vector times the magnitude of $\bar{w}_{C_a}$ as shown by

$$\bar{w}_{C_a} = \frac{(\bar{k}_{G_{a_2}} \times \bar{I}_3) \times \bar{k}_{G_{a_2}}}{\left|(\bar{k}_{G_{a_2}} \times \bar{I}_3) \times \bar{k}_{G_{a_2}}\right|}\, w_{C_a}.$$

As previously defined, the unit vector $\bar{k}_{G_{a_2}}(t)$ is parallel to the axle of the a gear wheel and $\bar{I}_3$ is perpendicular to the ground.

Therefore from the equation for $\bar{k}_{G_{a_2}}$ given in Appendix G it follows that

$$\bar{w}_{C_a} = \frac{(\bar{I}_3 - \beta_{G_{a_2}}{}^{b}\, \alpha_{G_{a_b}}{}^{c}\, \gamma_{B_c}{}^{3}\, \beta_{G_{a_2}}{}^{f}\, \alpha_{G_{a_f}}{}^{e}\, \gamma_{B_e}{}^{d}\, \bar{I}_d)}{D}\, w_{C_a},$$

where

$$D = [(\beta_{G_{a_2}}{}^{b}\, \alpha_{G_{a_b}}{}^{c}\, \gamma_{B_c}{}^{3}\, \beta_{G_{a_2}}{}^{f}\, \alpha_{G_{a_f}}{}^{e}\, \gamma_{B_e}{}^{1})^2$$

$$+ (\beta_{G_{a_2}}{}^{b}\, \alpha_{G_{a_b}}{}^{c}\, \gamma_{B_c}{}^{3}\, \beta_{G_{a_2}}{}^{f}\, \alpha_{G_{a_f}}{}^{e}\, \gamma_{B_e}{}^{2})^2$$

$$+ (1 - (\beta_{G_{a_2}}{}^{b}\, \alpha_{G_{a_b}}{}^{c}\, \gamma_{B_c}{}^{3})\ {}^{2})^2\ ]^{0.5},$$

and if each of $w_{C_a}{}^{b}$; b = 1, 3 is a simple graph such that if $t \geq 0$, then

$$\bar{w}_{C_a}(t) = w_{C_a}{}^{b}(t)\ \bar{I}_b$$

at the time t.

Thus, if $\bar{r}_{C_{F_a}}{}'$ is used for $\delta \bar{r}_{C_{F_a}}$ in the virtual work equation for the ground force and if each of $F_{G_a}{}^{b}$ is a simple graph such that if $t \geq 0$, $F_{G_a}{}^{b}(t)$ is the bth component of the a gear ground force such that at the time t

$$\bar{F}_{G_a}(t) = F_{G_a}{}^{b}(t)\ \bar{I}_b,$$

then after the indicated vector operations in equation (12) are performed, the quasi-coordinate and generalized coordinate force terms for the ground force are

$$s_G^{V_B}{}_f(t) = \sum_{a=1}^{N_G} \gamma_{B_f}{}^d(t)\,\delta_{bd}\,F_{G_a}{}^b(t) \qquad (f = 1, 3),$$

$$s_G^{\Omega_B}{}_c(t) = \sum_{a=1}^{N_G} (e_{cdh}\,\gamma_{B_f}{}^g(t)\,l_{T_a}{}^d\,\delta^{fh}\,F_{G_a}{}^b(t)\,\delta_{bg}$$

$$+ e_{cdh}\,\gamma_{B_f}{}^g(t)\,\alpha_{G_{a_e}}{}^d(t)\,(g_{A_a}{}^e + \xi_{A_a}{}^e(s_a(t)))\,\delta^{fh}\,F_{G_a}{}^b(t)\,\delta_{bg}$$

$$+ e_{fdb}\,\gamma_{B_c}{}^f(t)\,w_{C_a}{}^d(t)\,F_{G_a}{}^b(t) + e_{f3b}\,\gamma_{B_c}{}^f(t)\,v_{W_a}(t)\,F_{G_a}{}^b(t)) \quad (c = 1, 3),$$

$$s_G^{\Omega_{G_a}}(t) = \alpha_{G_{a_d}}{}^e(t)\,\gamma_{B_e}{}^g(t)\,e_{3hc}\,(g_{A_a}{}^h + \xi_{A_a}{}^h(s_a(t)))\,\delta^{cd}\,F_{G_a}{}^b(t)\,\delta_{bg}$$

$$+ e_{deb}\,\alpha_{G_{a_3}}{}^c(t)\,\gamma_{B_c}{}^d(t)\,w_{C_a}{}^e(t)\,F_{G_a}{}^b(t)$$

$$+ e_{d3b}\,\alpha_{G_{a_3}}{}^c(t)\,\gamma_{B_c}{}^d(t)\,v_{W_a}(t)\,F_{G_a}{}^b(t),$$

$$s_G^{\Omega_{W_a}}(t) = e_{cdb}\,\beta_{G_{a_2}}{}^p(t)\,\alpha_{G_{a_p}}{}^e(t)\,\gamma_{B_e}{}^c(t)\,w_{C_a}{}^d(t)\,F_{G_a}{}^b(t)$$

$$+ e_{c3b}\,\beta_{G_{a_2}}{}^p(t)\,\alpha_{G_{a_p}}{}^e(t)\,\gamma_{B_e}{}^c(t)\,v_{W_a}(t)\,F_{G_a}{}^b(t),$$

$$s_G^{q_B'}{}_f(t) = \sum_{a=1}^{N_G} \gamma_{G_{a_c}}{}^e\,\gamma_{B_e}{}^g(t)\,\Phi_{BG_{A_{a_f}}}{}^c(s_A(t), \theta_{G_a}(t))\,F_{G_a}{}^b(t)\,\delta_{bg}$$

$(f = 1, N_{BE})$,

$$s_G^{s_a'}(t) = \alpha_{G_{a_c}}{}^e(t)\,\gamma_{B_e}{}^g(t)\,\xi_{A_a}{}^{c\prime}(s_a(t))\,F_{G_a}{}^b(t)\,\delta_{bg},$$

$$S_{G}{}^{q_{G_a}{}'}{}_{f}(t) = \alpha_{G_{a_c}}{}^{e}(t)\ \gamma_{B_e}{}^{g}(t)\,\Phi_{G_{A_{a_f}}}{}^{c}(s_a(t),\ \theta_{G_a}(t))\ F_{G_a}{}^{b}(t)\ \delta_{bg}$$

$(f = 1,\ N_{GE_a})$,

where b ranges from 1 to 2.

To derive the ground force normal to the ground plane the following functions are needed:

Suppose that $\tilde{F}_{G_a}{}^{3}$ is the simple graph such that if $t \geq 0$, $\tilde{F}_{G_a}(x)$ is an ordinate of $\tilde{F}_{G_a}$ only if x is an ordinate of the simple graph v_{W_a} at the time t and $\tilde{F}_{G_a}{}^{3}(x)\ \bar{I}_3$ is the normal force from the ground on the a gear tire if $v_{W_a}(t)$ is equal to x.

Now suppose that $v_{W_{a_0}}$ is a number such that if x is a number greater than or equal to $v_{W_{a_0}}$ then $\tilde{F}_{G_a}{}^{3}(x) = 0$. Therefore, if u_a is a simple graph such that the point $(x,\ u_a(x))$ belongs to u_a only if x is an ordinate of v_{W_a} and x is less than $v_{W_{a_0}}$ and $u_a(x)$ is the potential energy stored in the a gear tire corresponding to x then

$$u_a(x) = -\int_{x}^{v_{W_{a_0}}} \tilde{F}_{G_a}{}^{3}\ dI,$$

or alternatively

$$u_a[v_{W_a}](t) = -\int_{v_{W_a}(t)}^{v_{W_{a_0}}} \tilde{F}_{G_a}^{\ 3}\, dI,$$

and since the generalized force associated with the coordinate v_{W_a} is defined by

$$S_G^{\ v_{W_a}{}'}(t) = -u_a{}'(v_{W_a}(t)),$$

it follows that

$$S_G^{\ v_{W_a}{}'}(t) = -\tilde{F}_{G_a}^{\ 3}(v_{W_a}(t)) = -F_{G_a}^{\ 3}(t).$$

The other two components of the ground force vector are derived in Appendix E.

Suppose that $\bar{V}_W$ is the velocity of the wind relative to the ground. Further suppose that α is a simple graph such that at the time t, $\alpha(t)$ is the angle of attack calculated from

$$\alpha(t) = \tan^{-1} \frac{(\bar{r}'(t) - \bar{V}_W) \cdot \bar{i}_{B_3}(t)}{(\bar{r}'(t) - \bar{V}_W) \cdot \bar{i}_{B_1}(t)}.$$

Also, suppose that β is a simple graph such that at the time t, $\beta(t)$ is the side slip angle calculated from

$$\beta(t) = \tan^{-1} \frac{(\bar{r}'(t) - \bar{V}_W) \cdot \bar{i}_{B_2}(t)}{(\bar{r}'(t) - \bar{V}_W) \cdot \bar{i}_{B_1}(t)}.$$

Further suppose that each of δ_e, δ_a and δ_r is the simple graph such that if $t \geq 0$, $\delta_e(t)$ is the elevator rotation, $\delta_a(t)$ is the aileron rotation and $\delta_r(t)$ is the rudder rotation at the time t.

Now if $\bar{F}_A$ is the vector function such that if $(x_B{}^1, x_B{}^2, x_B{}^3)$ is in V_B and $t \geq 0$, $\bar{F}_A(x_B{}^1, x_B{}^2, x_B{}^3, t)$ is the aerodynamic force per unit area for the point $(x_B{}^1, x_B{}^2, x_B{}^3)$ on the airframe surface at the time t, then the virtual work done by the aerodynamic force over the surface of the airframe S_B is

$$\delta W_A(t) = \int_{S_B} \bar{F}_A(I_{x_B}1, I_{x_B}2, I_{x_B}3, t) \cdot \delta \bar{R}_B(I_{x_B}1, I_{x_B}2, I_{x_B}3, t)\, dS_B.$$

Therefore, if q is a simple graph such that the time t, q(t) is the dynamic pressure and if ρ is the air density then

$$q(t) = 1/2\, \rho\, |\bar{r}'(t) - \bar{V}_W|^2,$$

and if S is the reference area for the aerodynamic forces and if it is assumed that the aerodynamic force is independent of the elastic deformation of the airframe then there is a vector function $\bar{\Lambda}_A$ such that

$$\bar{F}_A(x_B{}^1, x_B{}^2, x_B{}^3, t)$$
$$= q\, S\, \bar{\Lambda}_A\, (x_B{}^1, x_B{}^2, x_B{}^3, \alpha(t), \beta(t), \delta_e(t), \delta_a(t), \delta_r(t),$$
$$\Omega_B{}^1(t), \Omega_B{}^2(t), \Omega_B{}^3(t), t).$$

For the rigid body equations of motion the aerodynamic forces and moments are referenced to the point labeled P in the airframe. Therefore, if $\bar{I}_{NP}$ is the vector function such that if $t \geq 0$, $\bar{I}_{NP}(t)$ is the vector from

the jig condition location of the reference point N to the aerodynamic reference point P at the time t and if $\bar{L}_{PB}$ is the vector function such that if $(x_B^{\;1}, x_B^{\;2}, x_B^{\;3})$ is in V_B and $t \geq 0$, $\bar{L}_{PB}(x_B^{\;1}, x_B^{\;2}, x_B^{\;3}, t)$ is the vector from the jig condition location of the point P to the jig condition location of the point labeled $(x_B^{\;1}, x_B^{\;2}, x_B^{\;3})$ at the time t, it follows that

$$\bar{L}_B(x_B^{\;1}, x_B^{\;2}, x_B^{\;3}, t) = \bar{l}_{NP}(t) + \bar{L}_{PB}(x_B^{\;1}, x_B^{\;2}, x_B^{\;3}, t).$$

This expression may be used to modify equation (5) which is then used to replace $\delta\bar{R}_B$ in the virtual work equation. When the vector operations and integration are performed as indicated in the virtual work equation it is seen that the coefficients of the velocity terms may be expressed as

$$S_A{}^{V_B}{}_b(t) = F_P{}^c(t)\,\delta_{bc} \qquad (b = 1, 3),$$

$$S_A{}^{\Omega_B}{}_b(t) = M_P{}^c(t)\,\delta_{bc} + e_{bcd}\,F_P{}^d(t)\,l_{NP}{}^c \qquad (b = 1, 3),$$

$$S_A{}^{q_B{}'}{}_b(t) = Q_A{}^c(t)\,\delta_{bc} \qquad (b = 1, N_{BE}).$$

It is evident then that the aerodynamic force at the time t is $F_P{}^b(t)\,\bar{i}_{B_b}(t)$ and the aerodynamic moment at the time t is $\bar{i}_{B_b}(t)\,M_P{}^b(t)$. Further, it is seen that $Q_A{}^b(t)$ is the generalized force at the time t corresponding to the bth airframe vibration mode.

Now suppose that each of $C_F{}^b$; b= 1, 3 is a simple surface (aerodynamic force coefficient) such that the aerodynamic force component projection at

the time t on $\bar{i}_{B_b}(t)$ may be expressed by

$$F_P{}^b(t) =$$

$$q\ S\ C_F{}^b(\alpha(t), \beta(t), \delta_e(t), \delta_a(t), \delta_r(t), \Omega_B{}^1(t), \Omega_B{}^2(t), \Omega_B{}^3(t))$$

(b = 1, 3).

Also, suppose that each of $C_M{}^b$; b =1, 3 is a simple surface (an aerodynamic moment coefficient) and each of $l_A{}^b$; b = 1, 3 is an aerodynamic reference length such that the aerodynamic moment projection at the time t on $\bar{i}_{B_b}(t)$ may be expressed by

$$M_P{}^b(t) =$$

$$q\ S\ l_A{}^b\ C_M{}^b(\alpha(t), \beta(t), \delta_e(t), \delta_a(t), \delta_r(t), \Omega_B{}^1(t), \Omega_B{}^2(t), \Omega_B{}^3(t))$$

(b = 1, 3).

Further suppose that each of $C_Q{}^b$; b = 1, N_{BE} is a simple surface (an aerodynamic generalized force coefficient) such that

$$Q_A{}^b(t) =$$

$$q\ S\ C_Q{}^b(\alpha(t), \beta(t), \delta_e(t), \delta_a(t), \delta_r(t), \Omega_B{}^1(t), \Omega_B{}^2(t), \Omega_B{}^3(t))$$

(b = 1, N_{BE}).

The virtual work on the airframe from the thrust of the engine is assumed to be independent of the airframe deformation. That is, it is assumed that if TL is a point on the thrust line of the engine(s) then the airframe deformation in the direction of the thrust force is zero.

Suppose that $\bar{F}_T$ is the vector function such that if $t \geq 0$, $\bar{F}_T(t)$ is the thrust force vector at the time t and the magnitude of $\bar{F}_T(t)$ is a number. The thrust force vector $\bar{F}_T(t)$ may be expressed in component form as

$$\bar{F}_T(t) = F_T{}^b \, \bar{i}_{B_b}(t),$$

with $\bar{F}_T(t) \cdot \bar{i}_{B_2}(t) = 0$.

Also suppose that $\bar{l}_T$ is the vector function such that if $t \geq 0$, $\bar{l}_T(t)$ is the vector from the jig condition location of the airframe reference point N to the jig condition location of the point TL a the time t. Therefore, if $\bar{r}_{TL}$ is the vector function such that if $t \geq 0$, $\bar{r}_{TL}(t)$ is the vector from the ground reference point Q to the point TL at the time t, then the virtual work done by the thrust force is

$$\delta W_T(t) = \bar{F}_T(t) \cdot \delta \bar{r}_{TL}(t).$$

The vector $\bar{r}_{TL}(t)$ may be written in expanded form excluding the deformation terms as

$$\bar{r}_{TL}(t) = \bar{r}(t) + \bar{l}_T(t).$$

Thus,

$$\bar{r}_{TL}'(t) = \bar{r}'(t) + \bar{\Omega}_B(t) \times \bar{l}_T(t).$$

The vector $\bar{r}_{TL}'(t)$ may be used as a replacement for $\delta \bar{r}_{TL}(t)$ in the virtual work equation. It follows then that if in component form

$$\bar{l}_T(t) = l_T{}^b \, \bar{i}_{B_b}(t),$$

the quasi-coordinate force contributions from the thrust force are

$$S_T{}^{V_B}{}_b = F_T{}^c \, \delta_{bc} \qquad (b = 1, 3),$$

$$S_T{}^{\Omega_B}{}_2 = F_T{}^c \, e_{2bc} \, l_T{}^b .$$

Now suppose that $\bar{i}_{S_{a_3}}$ is the vector function such that if $t \geq 0$, $\bar{i}_{S_{a_3}}(t)$ is the unit vector at the time t which is parallel to the a gear shock strut axis of symmetry and directed downward with respect to the piston as shown in Figure B-2 of Appendix B. Further, suppose that $\bar{F}_{S_a}$ is the vector function such that if $t \geq 0$, $\bar{F}_{S_a}(t)$ is the shock strut force (see Appendix B for derivation of the shock strut force) at the time t. Also, suppose that F_{S_a} is the simple graph such that

$$\bar{F}_{S_a} = F_{S_a} \, \bar{i}_{S_{a_3}} .$$

Therefore, the selection of the a gear stroke, s_a, as a generalized coordinate permits the virtual work from the shock strut force to be expressed by

$$\delta W_S(t) = \sum_{a=1}^{N_G} F_{S_a}(t) \, \delta s_a(t).$$

Therefore, the contribution from the shock strut force to the generalized force in the a gear equation of motion is

$$S_S{}^{s_a{}'}(t) = F_{S_a}(t).$$

The deformation of the airframe and the deformation of the landing gears are assumed to be represented by normal vibration modes. Consequently, the stiffness matrices for the airframe and landing gear deformation equations of motion are diagonal. For simplicity, it is assumed that the damping is viscous and the damping matrices for these equations are diagonal. Based on these assumptions the following definitions are made.

Suppose that b is an integer in $[1, N_{BE}]$ and that the number $K_B^{\,b}$ is the generalized stiffness corresponding to the bth airframe modal equation. Further suppose that the number $C_B^{\,b}$ is the generalized damping corresponding to the bth airframe modal equation.

Therefore, the generalized force from the airframe modal stiffness and damping for the bth mode is

$$S_{BF}{}^{q_B{}'}{}_{b}(t) = - K_{B_b}\, q_B{}^{c}(t)\, \delta_{bc} - C_{B_b}\, q_B{}^{c}(t)\, \delta_{bc}.$$

Also, suppose that c is a positive integer in $[1, N_{GE_a}]$, s_{C_a} is the fully compressed stroke of the a gear and $K_{G_{a_c}}$ is a simple surface such that if x is in $[0, s_{C_a}]$, y is in $[-\pi, \pi]$, $K_{G_{a_c}}(x, y)$ is the generalized stiffness for the cth mode of the a gear corresponding to the a gear stroke x and castor angle y. Further, suppose that $C_{G_{a_c}}$ is a simple surface such that if x is in $[0, s_{C_a}]$, y is in $[-\pi, \pi]$, $C_{G_{a_c}}(x, y)$ is the generalized damping for the cth mode of the a gear corresponding to the a gear stroke x and castor angle y.

The generalized force from the a gear modal stiffness and damping for the cth mode is

$$S_{GF}{}^{q_{G_a}{}'}{}_{c}(t) = - K_{G_{a_c}}(s_a(t), \theta_{G_a}(t))\, q_{G_a}{}^{d}(t)\, \delta_{cd}$$

$$- C_{G_{a_c}}(s_a(t), \theta_{G_a}(t))\, q_{G_a}{}^{d}{}'(t)\, \delta_{cd}.$$

As stated previously, it is usual for only the nose gear to be capable of castoring. However, the final equations may be written more concisely if all the gears are assumed to have this degree of freedom.

The gear castoring moment is, in general, a nonlinear function of the castor angle function for the a gear, θ_{G_a}, and its derivative. The specific functional form of this moment is usually configuration dependent. Therefore, to simplify the current derivation it is assumed that M_C is a simple surface such that if $t \geq 0$, $M_C(\theta_{G_a}(t), \Omega_{G_a}(t))$ is the generalized moment associated with the generalized coordinate θ_{G_a} at the time t.

Thus, the castoring generalized force may be written as

$$S_{GC}{}^{G_a}(t) = - M_C(\theta_{G_a}(t), \Omega_{G_a}(t)).$$

The Constraint Condition

In the previous section the coordinate v_{W_a} was introduced when the ground forces acting on the system were defined. As a consequence there is a redundancy in the coordinates specifying the position of the system. This redundancy may be easily handled by use of a constraint condition and a Lagrangian multiplier as illustrated in Appendix C.

The constraint condition may be stated as follows: If at time t the a gear is on the ground then the component of the velocity $\bar{r}_{C_{F_a}}'(t)$, from equation (13), normal to the ground is zero. Therefore,

$$\bar{r}_{C_{F_a}}'(t) \cdot \bar{I}_3 = 0.$$

As seen from equation (13) this constraint condition has the form of equation (C-13) in Appendix C. Consequently, as seen from equation (C-14) the constraint function may be written in the following form:

$$\hat{F}_a[V_B{}^1,\ V_B{}^2,\ V_B{}^3,\ \Omega_B{}^1,\ \Omega_B{}^2,\ \Omega_B{}^3,\ \Omega_{G_a},$$

$$s_a',\ q_B{}^{1\prime},\ldots,\ q_B{}^{N_{BE}\prime},\ q_{G_a}{}^{1\prime},\ldots,\ q_{G_a}{}^{N_{GE_a}\prime},$$

$$\theta_{G_a},\ s_a,\ q_B{}^1,\ldots,q_B{}^{N_{BE}},\ q_{G_a}{}^1,\ldots,q_{G_a}{}^{N_{GE_a}},\ v_{W_a}'] = 0.$$

After the indicated vector operations have been performed, the terms to be used on the right side of equation (C-16) are

$$\hat{F}_{a;v_B}{}^{f} = \gamma_{B_f}{}^{3},$$

$$\hat{F}_{a;\Omega_B}{}^{c} = e_{cdh}\, \gamma_{B_f}{}^{3}\, {}^{1}T_a{}^{d}\, \delta^{fh}$$

$$+ e_{cdh}\, \gamma_{B_f}{}^{3}\, \alpha_{G_{a_c}}{}^{d}\, (g_{A_a}{}^{c} + \xi_{A_a}{}^{c}[s_a])\, \delta^{fh}$$

$$+ e_{fd3}\, \gamma_{B_c}{}^{f}\, w_{C_a}{}^{d},$$

$$\hat{F}_{a;\Omega_{G_a}} = \alpha_{G_{a_d}}{}^{e}\, \gamma_{B_e}{}^{3}\, e_{3hc}\, (g_{A_a}{}^{h} + \xi_{A_a}{}^{h}[\ _a])\, \delta^{cd}$$

$$+ e_{de3}\, \alpha_{G_{a_3}}{}^{c}\, \gamma_{B_c}{}^{d}\, w_{C_a}{}^{e},$$

$$\hat{F}_{a;\Omega_{W_a}} = e_{cd3}\, \beta_{G_{a_2}}{}^{p}\, \alpha_{G_{a_p}}{}^{e}\, \gamma_{B_e}{}^{c}\, w_{C_a}{}^{d},$$

$$\hat{F}_{a;q_B}{}^{f}{}' = \gamma_{G_{a_c}}{}^{e}\, \gamma_{B_e}{}^{3}\, \Phi_{BG_{A_{a_f}}}{}^{c}[s_a, \theta_{G_a}],$$

$$\hat{F}_{a;s_a}{}' = \alpha_{G_{a_c}}{}^{e}\, \gamma_{B_e}{}^{3}\, \xi_{A_a}{}^{c}{}'[s_a],$$

$$\hat{F}_{a;q_{G_a}}{}^{f}{}' = \alpha_{G_{a_c}}{}^{e}\, \gamma_{B_e}{}^{3}\, \Phi_{G_{A_{a_f}}}{}^{c}[s_a, \theta_{G_a}],$$

and

$$\hat{F}_{a;v_{W_a}}{}' = 1.0.$$

Equations of Motion

All of the input terms to the equations of motion (equation (C-16) of Appendix C) have now been derived. The definitions

$$S^{V_B}_{\ b} = S_W{}^{V_B}_{\ b} + S_H{}^{V_B}_{\ b} + S_G{}^{V_B}_{\ b} + S_A{}^{V_B}_{\ b} + S_T{}^{V_B}_{\ b} \qquad (b = 1,\ 3),$$

$$S^{\Omega_B}_{\ \ b} = S_W{}^{\Omega_B}_{\ \ b} + S_H{}^{\Omega_B}_{\ \ b} + S_G{}^{\Omega_B}_{\ \ b} + S_A{}^{\Omega_B}_{\ \ b} + S_T{}^{\Omega_B}_{\ \ b} \qquad (b = 1,\ 3),$$

$$S^{\Omega_{G_a}} = S_W{}^{\Omega_{G_a}} + S_G{}^{\Omega_{G_a}} + S_{GC}{}^{\Omega_{G_a}},$$

$$S^{\Omega_{W_a}} = S_G{}^{\Omega_{W_a}},$$

$$S^{s_a{}'} = S_W{}^{s_a{}'} + S_G{}^{s_a{}'} + S_S{}^{s_a{}'},$$

$$S^{q_B{}'}_{\ \ b} = S_W{}^{q_B{}'}_{\ \ b} + S_H{}^{q_B{}'}_{\ \ b} + S_G{}^{q_B{}'}_{\ \ b} + S_A{}^{q_B{}'}_{\ \ b} + S_{BF}{}^{q_B{}'}_{\ \ b} \qquad (b = 1,\ N_{BE}),$$

$$S^{q_{G_a}{}'}_{\ \ \ b} = S_W{}^{q_{G_a}{}'}_{\ \ \ b} + S_G{}^{q_{G_a}{}'}_{\ \ \ b} + S_{GF}{}^{q_{G_a}{}'}_{\ \ \ b} \qquad (b = 1,\ N_{GE_a}),$$

permit the equations of motion to be expressed in the following form:

$$0 = S_G{}^{v_{W_a}} + \lambda_a \hat{F}_{a;v_{W_a}},$$

$$(K_{;V_B}{}^1)' - \Omega_B{}^3 K_{;V_B}{}^2 + \Omega_B{}^2 K_{;V_B}{}^3 = S^{V_B}_{\ \ 1} + \sum_{a=1}^{N_G} \lambda_a \hat{F}_{a;V_B}{}^1.$$

$$(K_{;V_B}{}^2)' + \Omega_B{}^3 K_{;V_B}{}^1 - \Omega_B{}^1 K_{;V_B}{}^3 = S^{V_B}{}_2 + \sum_{a=1}^{N_G} \lambda_a \hat{F}_{a;V_B}{}^2.$$

$$(K_{;V_B}{}^3)' - \Omega_B{}^2 K_{;V_B}{}^1 + \Omega_B{}^1 K_{;V_B}{}^2 = S^{V_B}{}_3 + \sum_{a=1}^{N_G} \lambda_a \hat{F}_{a;V_B}{}^3.$$

$$(K_{;\Omega_B}{}^1)' - V_B{}^3 K_{;V_B}{}^2 + V_B{}^2 K_{;V_B}{}^3 - \Omega_B{}^3 K_{;\Omega_B}{}^2 + \Omega_B{}^2 K_{;\Omega_B}{}^3 =$$
$$S^{\Omega_B}{}_1 + \sum_{a=1}^{N_G} \lambda_a \hat{F}_{a;\Omega_B}{}^1.$$

$$(K_{;\Omega_B}{}^2)' + V_B{}^3 K_{;V_B}{}^1 - V_B{}^1 K_{;V_B}{}^3 + \Omega_B{}^3 K_{;\Omega_B}{}^1 - \Omega_B{}^1 K_{;\Omega_B}{}^3 =$$
$$S^{\Omega_B}{}_2 + \sum_{a=1}^{N_G} \lambda_a \hat{F}_{;\Omega_B}{}^2.$$

$$(K_{;\Omega_B}{}^3)' - V_B{}^2 K_{;V_B}{}^1 + V_B{}^1 K_{;V_B}{}^2 - \Omega_B{}^2 K_{;\Omega_B}{}^1 + \Omega_B{}^1 K_{;\Omega_B}{}^2 =$$
$$S^{\Omega_B}{}_3 + \sum_{a=1}^{N_G} \lambda_a \hat{F}_{;\Omega_B}{}^3.$$

$$(K_{;\Omega_{G_a}})' - K_{;\Omega_{G_a}} = S^{\Omega_{G_a}} + \lambda_a \hat{F}_{;\Omega_{G_a}} \qquad (a = 1, N_G).$$

$$(K_{;\Omega_{W_a}})' = S^{\Omega_{W_a}} \qquad (a = 1, N_G).$$

$$(K_{;s_a{}'})' - K_{;s_a} = S^{s_a{}'} + \lambda_a \hat{F}_{;s_a{}'} \qquad (a = 1, N_G).$$

$$(K_{;q_B{}^{b}{}'})' - K_{;q_B{}^{b}} = S^{q_B{}'}{}_{b} + \sum_{a=1}^{N_G} \lambda_a \hat{F}_{;q_B{}^{b}}, \qquad (b = 1, N_{BE}).$$

$$(K_{;q_{G_a}{}^{b}{}'})' - K_{;q_{G_a}{}^{b}} = S^{q_{G_a}{}'}{}_{b} + \lambda_a \hat{F}_{a;q_{G_a}{}^{b}}, \qquad (a = 1, N_G \text{ and } b = 1, N_{GE_a}).$$

These equations may be rewritten by use of the expanded expression for the kinetic energy and the generalized and quasi-coordinate forces.

The first equation of motion above permits the calculation of the Lagrangian multiplier λ_a. Since

$$\hat{F}_{a;v_{W_a}{}'} = 1.0$$

and

$$S_G{}^{v_{W_a}{}'} = -F_{G_a}{}^{3},$$

then

$$\lambda_a = F_{G_a}{}^{3}.$$

Thus, the Lagrangian multiplier λ_a is seen to be the component of the force on the a gear tire from the ground that is normal to the ground.

The V_B^{1} equation is

$$
\begin{aligned}
&(M^{V_B V_B}_{1\;\;1})\, V_B^{1\prime} + (M^{V_B \Omega_B}_{1\;\;c})[s_a, \theta_{G_a}; a = 1, N_G]\, \Omega_B^{c\prime} \\
&+ \sum_{a=1}^{N_G} (M^{V_B \Omega_B}_{1\;\;c})_{;s_a} s_a' \, \Omega_B^{c} + \sum_{a=1}^{N_G} (M^{V_B \Omega_B}_{1\;\;c})_{;\theta_{G_a}} \Omega_{G_a} \Omega_B^{c} \\
&+ \sum_{a=1}^{N_G} (M^{V_B \Omega_{G_a}}_{1})[s_a, \theta_{G_a}]\, \Omega_{G_a}' + \sum_{a=1}^{N_G} (M^{V_B \Omega_{G_a}}_{1})_{;s_a} s_a' \, \Omega_{G_a} \\
&+ \sum_{a=1}^{N_G} (M^{V_B \Omega_{G_a}}_{1})_{;\theta_{G_a}} (\Omega_{G_a})^2 \\
&+ \sum_{a=1}^{N_G} (M^{V_B s_a}_{1})[s_a, \theta_{G_a}]\, s_a'' + \sum_{a=1}^{N_G} (M^{V_B s_a}_{1})_{;s_a} (s_a')^2 \\
&+ \sum_{a=1}^{N_G} (M^{V_B s_a}_{1})_{;\theta_{G_a}} \Omega_{G_a} s_a' + (M^{V_B q_B}_{1\;\;c})[s_a, \theta_{G_a}; a = 1, N_G]\, q_B^{c\prime\prime} \\
&+ \sum_{a=1}^{N_G} (M^{V_B q_B}_{1\;\;c})_{;s_a} s_a' \, q_B^{c\prime} + \sum_{a=1}^{N_G} (M^{V_B q_B}_{1\;\;c})_{;\theta_{G_a}} \Omega_{G_a} q_B^{c\prime} \\
&+ \sum_{a=1}^{N_G} (M^{V_B q_{G_a}}_{1\;\;c})[s_a, \theta_{G_a}]\, q_{G_a}^{c\prime\prime} + \sum_{a=1}^{N_G} (M^{V_B q_{G_a}}_{1\;\;c})_{;s_a} s_a' \, q_{G_a}^{c\prime} \\
&+ \sum_{a=1}^{N_G} (M^{V_B q_{G_a}}_{1\;\;c})_{;\theta_{G_a}} \Omega_{G_a} q_{G_a}^{c\prime}
\end{aligned}
\tag{14}
$$

$$-\Omega_B{}^3 \left(\left(M^{V_B V_B}_{1\ 1}\right) V_B{}^2 + \left(M^{V_B \Omega_B}_{2\ \ c}\right)[s_a, \theta_{G_a}; a = 1, N_G]\, \Omega_B{}^c\right.$$

$$+ \sum_{a=1}^{N_G} \left(M^{V_B \Omega_{G_a}}_{2}\right)[s_a, \theta_{G_a}]\, \Omega_{G_a}$$

$$+ \sum_{a=1}^{N_G} \left(M^{V_B s_a}_{2}\right)[s_a, \theta_{G_a}]\, s_a{}' + \left(M^{V_B q_B}_{2\ \ c}\right)[s_a, \theta_{G_a}; a = 1, N_G]\, q_B{}^{c\prime}$$

$$\left. + \sum_{a=1}^{N_G} \left(M^{V_B q_{G_a}}_{2\ \ c}\right)[s_a, \theta_{G_a}]\, q_{G_a}{}^{c\prime}\right)$$

$$+\Omega_B{}^2 \left(\left(M^{V_B V_B}_{1\ 1}\right) V_B{}^3 + \left(M^{V_B \Omega_B}_{3\ \ c}\right)[s_a, \theta_{G_a}; a = 1, N_G]\, \Omega_B{}^c\right.$$

$$+ \sum_{a=1}^{N_G} \left(M^{V_B \Omega_{G_a}}_{3}\right)[s_a, \theta_{G_a}]\, \Omega_{G_a}$$

$$+ \sum_{a=1}^{N_G} \left(M^{V_B s_a}_{3}\right)[s_a, \theta_{G_a}]\, s_a{}' + \left(M^{V_B q_B}_{3\ \ c}\right)[s_a, \theta_{G_a}; a = 1, N_G]\, q_B{}^{c\prime}$$

$$\left. + \sum_{a=1}^{N_G} \left(M^{V_B q_{G_a}}_{3\ \ c}\right)[s_a, \theta_{G_a}]\, q_{G_a}{}^{c\prime}\right)$$

$$= \sum_{a=1}^{N_G} F_{G_a}{}^b \left(\gamma_{B_1}^d\, \delta_{bd}\right) + g \left(M^{V_B V_B}_{1\ 1}\right) \gamma_{B_1}^3 + F_P{}^b\, \delta_{1b}$$

$$+ F_H{}^b \left(\gamma_{B_1}^d\, \delta_{bd}\right) + F_T{}^b\, \delta_{1b}.$$

The V_B^2 equation is

$$(M^{V_B V_B}_{1\ 1})\, V_B^{2\prime} + (M^{V_B \Omega_B}_{2\ c})[s_a, \theta_{G_a}; a = 1, N_G]\, \Omega_B^{c\prime} \tag{15}$$

$$+ \sum_{a=1}^{N_G} (M^{V_B \Omega_B}_{2\ c})_{;s_a} s_a' \, \Omega_B^{c} + \sum_{a=1}^{N_G} (M^{V_B \Omega_B}_{2\ c})_{;\theta_{G_a}} \Omega_{G_a} \Omega_B^{c}$$

$$+ \sum_{a=1}^{N_G} (M^{V_B \Omega_{G_a}}_{2})[s_a, \theta_{G_a}]\, \Omega_{G_a}' + \sum_{a=1}^{N_G} (M^{V_B \Omega_{G_a}}_{2})_{;s_a} s_a' \, \Omega_{G_a}$$

$$+ \sum_{a=1}^{N_G} (M^{V_B \Omega_{G_a}}_{2})_{;\theta_{G_a}} (\Omega_{G_a})^2$$

$$+ \sum_{a=1}^{N_G} (M^{V_B s_a}_{2})[s_a, \theta_{G_a}]\, s_a'' + \sum_{a=1}^{N_G} (M^{V_B s_a}_{2})_{;s_a} (s_a')^2$$

$$+ \sum_{a=1}^{N_G} (M^{V_B s_a}_{2})_{;\theta_{G_a}} \Omega_{G_a} s_a' + (M^{V_B q_B}_{2\ c})[s_a, \theta_{G_a}; a = 1, N_G]\, q_B^{c\prime\prime}$$

$$+ \sum_{a=1}^{N_G} (M^{V_B q_B}_{2\ c})_{;s_a} s_a' \, q_B^{c\prime} + \sum_{a=1}^{N_G} (M^{V_B q_B}_{2\ c})_{;\theta_{G_a}} \Omega_{G_a} q_B^{c\prime}$$

$$+ \sum_{a=1}^{N_G} (M^{V_B q_{G_a}}_{2\ c})[s_a, \theta_{G_a}]\, q_{G_a}^{c\prime\prime} + \sum_{a=1}^{N_G} (M^{V_B q_{G_a}}_{2\ c})_{;s_a} s_a' \, q_{G_a}^{c\prime}$$

$$+ \sum_{a=1}^{N_G} (M^{V_B q_{G_a}}_{2\ c})_{;\theta_{G_a}} \Omega_{G_a} q_{G_a}^{c\prime}$$

$$+\Omega_B{}^3 \, ((M^{V_B V_B}{}_{1\,1}) \, V_B{}^1 + (M^{V_B \Omega_B}{}_{1\,c})[s_a, \theta_{G_a}; a = 1, N_G] \, \Omega_B{}^c$$

$$+\sum_{a=1}^{N_G} (M^{V_B \Omega_{G_a}}{}_{1})[s_a, \theta_{G_a}] \, \Omega_{G_a}$$

$$+\sum_{a=1}^{N_G} (M^{V_B s_a}{}_{1})[s_a, \theta_{G_a}] \, s_a{}' + (M^{V_B q_B}{}_{1\,c})[s_a, \theta_{G_a}; a = 1, N_G] \, q_B{}^{c\prime}$$

$$+\sum_{a=1}^{N_G} (M^{V_B q_{G_a}}{}_{1\,c})[s_a, \theta_{G_a}] \, q_{G_a}{}^{c\prime})$$

$$-\Omega_B{}^1 \, ((M^{V_B V_B}{}_{1\,1}) \, V_B{}^3 + (M^{V_B \Omega_B}{}_{3\,c})[s_a, \theta_{G_a}; a = 1, N_G] \, \Omega_B{}^c$$

$$+\sum_{a=1}^{N_G} (M^{V_B \Omega_{G_a}}{}_{3})[s_a, \theta_{G_a}] \, \Omega_{G_a}$$

$$+\sum_{a=1}^{N_G} (M^{V_B s_a}{}_{3})[s_a, \theta_{G_a}] \, s_a{}' + (M^{V_B q_B}{}_{3\,c})[s_a, \theta_{G_a}; a = 1, N_G] \, q_B{}^{c\prime}$$

$$+\sum_{a=1}^{N_G} (M^{V_B q_{G_a}}{}_{3\,c})[s_a, \theta_{G_a}] \, q_{G_a}{}^{c\prime})$$

$$= \sum_{a=1}^{N_G} F_{G_a}{}^b \, (\gamma_{B_2}{}^d \, \delta_{bd}) + g \, (M^{V_B V_B}{}_{1\,1}) \, \gamma_{B_2}{}^3 + F_P{}^b \, \delta_{2b}$$

$$+ F_H{}^b \, (\gamma_{B_2}{}^d \, \delta_{bd}).$$

The $V_B^{\,3}$ equation is

$$
\begin{aligned}
&(M^{V_B V_B}_{1\;1})\, V_B^{3\prime} + (M^{V_B \Omega_B}_{3\;c})[s_a, \theta_{G_a}; a = 1, N_G]\, \Omega_B^{c\prime} \\
&+ \sum_{a=1}^{N_G} (M^{V_B \Omega_B}_{3\;c})_{;s_a}\, s_a'\, \Omega_B^{c} + \sum_{a=1}^{N_G} (M^{V_B \Omega_B}_{3\;c})_{;\theta_{G_a}}\, \Omega_{G_a} \Omega_B^{c} \\
&+ \sum_{a=1}^{N_G} (M^{V_B \Omega_{G_a}}_{3})[s_a, \theta_{G_a}]\, \Omega_{G_a}' + \sum_{a=1}^{N_G} (M^{V_B \Omega_{G_a}}_{3})_{;s_a}\, s_a'\, \Omega_{G_a} \\
&+ \sum_{a=1}^{N_G} (M^{V_B \Omega_{G_a}}_{3})_{;\theta_{G_a}}\, (\Omega_{G_a})^2 \\
&+ \sum_{a=1}^{N_G} (M^{V_B s_a}_{3})[s_a, \theta_{G_a}]\, s_a'' + \sum_{a=1}^{N_G} (M^{V_B s_a}_{3})_{;s_a}\, (s_a')^2 \\
&+ \sum_{a=1}^{N_G} (M^{V_B s_a}_{3})_{;\theta_{G_a}}\, \Omega_{G_a}\, s_a' + (M^{V_B q_B}_{3\;c})[s_a, \theta_{G_a}; a = 1, N_G]\, q_B^{c\prime\prime} \\
&+ \sum_{a=1}^{N_G} (M^{V_B q_B}_{3\;c})_{;s_a}\, s_a'\, q_B^{c\prime} + \sum_{a=1}^{N_G} (M^{V_B q_B}_{3\;c})_{;\theta_{G_a}}\, \Omega_{G_a}\, q_B^{c\prime} \\
&+ \sum_{a=1}^{N_G} (M^{V_B q_{G_a}}_{3\;c})[s_a, \theta_{G_a}]\, q_{G_a}^{c\prime\prime} + \sum_{a=1}^{N_G} (M^{V_B q_{G_a}}_{3\;c})_{;s_a}\, s_a'\, q_{G_a}^{c\prime} \\
&+ \sum_{a=1}^{N_G} (M^{V_B q_{G_a}}_{3\;c})_{;\theta_{G_a}}\, \Omega_{G_a}\, q_{G_a}^{c\prime}
\end{aligned}
\tag{16}
$$

$$-\Omega_B^{\,2} \left((M^{V_B V_B}_{1\ \ 1}) V_B^{\,1} + (M^{V_B \Omega_B}_{1\ \ c})[s_a, \theta_{G_a}; a = 1, N_G] \Omega_B^{\,c} \right.$$

$$+ \sum_{a=1}^{N_G} (M^{V_B \Omega_{G_a}}_{1})[s_a, \theta_{G_a}] \Omega_{G_a}$$

$$+ \sum_{a=1}^{N_G} (M^{V_B s_a}_{1})[s_a, \theta_{G_a}] s_a' + (M^{V_B q_B}_{1\ \ c})[s_a, \theta_{G_a}; a = 1, N_G] q_B^{\,c\prime}$$

$$\left. + \sum_{a=1}^{N_G} (M^{V_B q_{G_a}}_{1\ \ c})[s_a, \theta_{G_a}] q_{G_a}^{\,c\prime} \right)$$

$$+ \Omega_B^{\,1} \left((M^{V_B V_B}_{1\ \ 1}) V_B^{\,2} + (M^{V_B \Omega_B}_{2\ \ c})[s_a, \theta_{G_a}; a = 1, N_G] \Omega_B^{\,c} \right.$$

$$+ \sum_{a=1}^{N_G} (M^{V_B \Omega_{G_a}}_{2})[s_a, \theta_{G_a}] \Omega_{G_a}$$

$$+ \sum_{a=1}^{N_G} (M^{V_B s_a}_{2})[s_a, \theta_{G_a}] s_a' + (M^{V_B q_B}_{2\ \ c})[s_a, \theta_{G_a}; a = 1, N_G] q_B^{\,c\prime}$$

$$\left. + \sum_{a=1}^{N_G} (M^{V_B q_{G_a}}_{2\ \ c})[s_a, \theta_{G_a}] q_{G_a}^{\,c\prime} \right)$$

$$= \sum_{a=1}^{N_G} F_{G_a}^{\,b} (\gamma_{B_3}^{\,d} \delta_{bd}) + g\,(M^{V_B V_B}_{1\ \ 1}) \gamma_{B_3}^{\,3} + F_P^{\,b} \delta_{3b}$$

$$+ F_H^{\,b} (\gamma_{B_3}^{\,d} \delta_{bd}) + F_T^{\,b} \delta_{3b}.$$

The $\Omega_B{}^1$ equation is

$$
\begin{aligned}
&(M^{\Omega_B\Omega_B}_{1\;c})[s_a,\theta_{G_a};\ a=1,N_G]\,\Omega_B{}^{c\prime} + \sum_{a=1}^{N_G}(M^{\Omega_B\Omega_B}_{1\;c})_{;s_a}\,\Omega_B{}^{c}\,s_a{}' \\
&+ \sum_{a=1}^{N_G}(M^{\Omega_B\Omega_B}_{1\;c})_{;\theta_{G_a}}\,\Omega_B{}^{c}\,\Omega_{G_a} + (M^{V_B\Omega_B}_{b\;1})[s_a,\theta_{G_a};\ a=1,N_G]\,V_B{}^{b\prime} \\
&+ \sum_{a=1}^{N_G}(M^{V_B\Omega_B}_{b\;1})_{;s_a}\,s_a{}'\,V_B{}^{b\prime} + \sum_{a=1}^{N_G}(M^{V_B\Omega_B}_{b\;1})_{;\theta_{G_a}}\,\Omega_{G_a}\,V_B{}^{b\prime} \\
&+ \sum_{a=1}^{N_G}(M^{\Omega_B\Omega_{G_a}}_{1})[s_a,\theta_{G_a}]\,\Omega_{G_a}{}' + \sum_{a=1}^{N_G}(M^{\Omega_B\Omega_{G_a}}_{1})_{;s_a}\,s_a{}'\,\Omega_{G_a} \\
&+ \sum_{a=1}^{N_G}(M^{\Omega_B\Omega_{G_a}}_{1})_{;\theta_{G_a}}\,(\Omega_{G_a})^2 + \sum_{a=1}^{N_G}(M^{\Omega_B\Omega_{W_a}}_{1})[s_a,\theta_{G_a}]\,\Omega_{W_a}{}' \\
&+ \sum_{a=1}^{N_G}(M^{\Omega_B\Omega_{W_a}}_{1})_{;s_a}\,s_a{}'\,\Omega_{W_a} + \sum_{a=1}^{N_G}(M^{\Omega_B\Omega_{W_a}}_{1})_{;\theta_{G_a}}\,\Omega_{G_a}\,\Omega_{W_a} \\
&+ \sum_{a=1}^{N_G}(M^{\Omega_B s_a}_{1})[s_a,\theta_{G_a}]\,s_a{}'' + \sum_{a=1}^{N_G}(M^{\Omega_B s_a}_{1})_{;s_a}\,(s_a{}')^2 \\
&+ \sum_{a=1}^{N_G}(M^{\Omega_B s_a}_{1})_{;\theta_{G_a}}\,\Omega_{G_a}\,s_a{}' + (M^{\Omega_B q_B}_{1\;c})[s_a,\theta_{G_a};\ a=1,N_G]\,q_B{}^{c\prime\prime} \\
&+ \sum_{a=1}^{N_G}(M^{\Omega_B q_B}_{1\;c})_{;s_a}\,s_a{}'\,q_B{}^{c\prime} + \sum_{a=1}^{N_G}(M^{\Omega_B q_B}_{1\;c})_{;\theta_{G_a}}\,\Omega_{G_a}\,q_B{}^{c\prime} \\
&+ \sum_{a=1}^{N_G}(M^{\Omega_B q_{G_a}}_{1\;c})[s_a,\theta_{G_a}]\,q_{G_a}{}^{c\prime\prime} + \sum_{a=1}^{N_G}(M^{\Omega_B q_{G_a}}_{1\;c})_{;s_a}\,s_a{}'\,q_{G_a}{}^{c\prime} \\
&+ \sum_{a=1}^{N_G}(M^{\Omega_B q_{G_a}}_{1\;c})_{;\theta_{G_a}}\,\Omega_{G_a}\,q_{G_a}{}^{c\prime}
\end{aligned}
\tag{17}
$$

$$- V_B^{\,3} \, ((M^{V_B \Omega_B}{}_{2\ c})[s_a, \theta_{G_a}; a = 1, N_G] \, \Omega_B^{\ c}$$

$$+ \sum_{a=1}^{N_G} (M^{V_B \Omega_{G_a}}{}_{2})[s_a, \theta_{G_a}] \, \Omega_{G_a} + \sum_{a=1}^{N_G} (M^{V_B s_a}{}_{2})[s_a, \theta_{G_a}] \, s_a'$$

$$+ (M^{V_B q_B}{}_{2\ c})[s_a, \theta_{G_a}; a = 1, N_G] \, q_B^{\ c\prime} + \sum_{a=1}^{N_G} (M^{V_B q_{G_a}}{}_{2\ c})[s_a, \theta_{G_a}] \, q_{G_a}^{\ c\prime})$$

$$+ V_B^{\,2} \, ((M^{V_B \Omega_B}{}_{3\ c})[s_a, \theta_{G_a}; a = 1, N_G] \, \Omega_B^{\ c}$$

$$+ \sum_{a=1}^{N_G} (M^{V_B \Omega_{G_a}}{}_{3})[s_a, \theta_{G_a}] \, \Omega_{G_a} + \sum_{a=1}^{N_G} (M^{V_B s_a}{}_{3})[s_a, \theta_{G_a}] \, s_a'$$

$$+ (M^{V_B q_B}{}_{3\ c})[s_a, \theta_{G_a}; a = 1, N_G] \, q_B^{\ c\prime} + \sum_{a=1}^{N_G} (M^{V_B q_{G_a}}{}_{3\ c})[s_a, \theta_{G_a}] \, q_{G_a}^{\ c\prime})$$

$$- \Omega_B^{\,3} \, ((M^{\Omega_B \Omega_B}{}_{2\ c})[s_a, \theta_{G_a}; a = 1, N_G] \, \Omega_B^{\ c}$$

$$+ (M^{V_B \Omega_B}{}_{b\ 2})[s_a, \theta_{G_a}; a = 1, N_G] \, V_B^{\ b}$$

$$+ \sum_{a=1}^{N_G} (M^{\Omega_B \Omega_{G_a}}{}_{2})[s_a, \theta_{G_a}] \, \Omega_{G_a} + \sum_{a=1}^{N_G} (M^{\Omega_B \Omega_{W_a}}{}_{2})[s_a, \theta_{G_a}] \, \Omega_{W_a}$$

$$+ \sum_{a=1}^{N_G} (M^{\Omega_B s_a}{}_{2})[s_a, \theta_{G_a}] \, s_a' + (M^{\Omega_B q_B}{}_{2\ c})[s_a, \theta_{G_a}; a = 1, N_G] \, q_B^{\ c\prime}$$

$$+ \sum_{a=1}^{N_G} (M^{\Omega_B q_{G_a}}{}_{2\ c})[s_a, \theta_{G_a}] \, q_{G_a}')$$

$$+\Omega_B{}^2\,\big((M^{\Omega_B\Omega_B}{}_{3c})[s_a,\ \theta_{G_a};\ a = 1,\ N_G]\,\Omega_B{}^c$$

$$+\ (M^{v_B\Omega_B}{}_{b3})[s_a,\ \theta_{G_a};\ a = 1,\ N_G]\ v_B{}^b$$

$$+\sum_{a=1}^{N_G}(M^{\Omega_B\Omega_{G_a}}{}_3)[s_a,\ \theta_{G_a}]\,\Omega_{G_a} + \sum_{a=1}^{N_G}(M^{\Omega_B\Omega_{W_a}}{}_3)[s_a,\ \theta_{G_a}]\,\Omega_{W_a}$$

$$+\sum_{a=1}^{N_G}(M^{\Omega_B s_a}{}_3)[s_a,\ \theta_{G_a}]\ s_a{}' + (M^{\Omega_B q_B}{}_{3c})[s_a,\ \theta_{G_a};\ a = 1,\ N_G]\ q_B{}^{c\prime}$$

$$+\sum_{a=1}^{N_G}(M^{\Omega_B q_{G_a}}{}_{3c})[s_a,\ \theta_{G_a}]\ q_{G_a}{}'\big)$$

$$= \sum_{a=1}^{N_G} F_{G_a}{}^b\,\big(e_{1dh}\,\gamma_{B_f}^{g}\,1_{T_a}{}^d\,\delta_{bg}\,\delta^{fh}$$

$$+\ e_{1dh}\,\gamma_{B_f}^{g}\,\alpha_{{G_a}_e}{}^d\,(g_{A_a}{}^e + \xi_{A_a}{}^e[s_a])\,\delta_{bg}\,\delta^{fh}$$

$$+\ e_{fdb}\,\gamma_{B_1}^{f}\,w_{C_a}{}^d + e_{f3b}\,\gamma_{B_1}^{f}\,v_{W_a}\big)$$

$$+\ g\,(M^{v_B\Omega_B}{}_{c1})[s_a,\ \theta_{G_a};\ a = 1,\ N_G]\ \gamma_{B_d}^{3}\,\delta^{cd}$$

$$+\ F_H{}^b\,\big(e_{1dh}\,\gamma_{B_f}^{g}\,1_H{}^d\,\delta_{bg}\,\delta^{fh}$$

$$+\ (\gamma_{B_1}^{g}\,\alpha_{H_3}{}^h\,\gamma_{H_h}^{p}\,\delta_{pg})\,e_{sdr}$$

$$\times\ \alpha_{H_3}{}^f\,\gamma_{H_f}^{d}\,(a_1\,\gamma_{B_1}^{r} + b_1\,\gamma_{B_3}^{r})\,\delta_b^s\big) + F_P{}^d\,e_{1cd}\,1_{NP}{}^c + M_P{}^b\,\delta_{1b}.$$

The $\Omega_B^{\,2}$ equation is

$$
\begin{aligned}
&(M^{\Omega_B \Omega_B}_{2\ c})[s_a, \theta_{G_a};\ a = 1, N_G]\, \Omega_B^{c\prime} + \sum_{a=1}^{N_G} (M^{\Omega_B \Omega_B}_{2\ c})_{;s_a} \Omega_B^{c}\, s_a' \qquad (18) \\
&+ \sum_{a=1}^{N_G} (M^{\Omega_B \Omega_B}_{2\ c})_{;\theta_{G_a}} \Omega_B^{c}\, \Omega_{G_a} + (M^{V_B \Omega_B}_{b\ 2})[s_a, \theta_{G_a};\ a = 1, N_G]\, V_B^{b\prime} \\
&+ \sum_{a=1}^{N_G} (M^{V_B \Omega_B}_{b\ 2})_{;s_a} s_a'\, V_B^{b\prime} + \sum_{a=1}^{N_G} (M^{V_B \Omega_B}_{b\ 2})_{;\theta_{G_a}} \Omega_{G_a}\, V_B^{b\prime} \\
&+ \sum_{a=1}^{N_G} (M^{\Omega_B \Omega_{G_a}}_{2})[s_a, \theta_{G_a}]\, \Omega_{G_a}' + \sum_{a=1}^{N_G} (M^{\Omega_B \Omega_{G_a}}_{2})_{;s_a} s_a'\, \Omega_{G_a} \\
&+ \sum_{a=1}^{N_G} (M^{\Omega_B \Omega_{G_a}}_{2})_{;\theta_{G_a}} (\Omega_{G_a})^2 + \sum_{a=1}^{N_G} (M^{\Omega_B \Omega_{W_a}}_{2})[s_a, \theta_{G_a}]\, \Omega_{W_a}' \\
&+ \sum_{a=1}^{N_G} (M^{\Omega_B \Omega_{W_a}}_{2})_{;s_a} s_a'\, \Omega_{W_a} + \sum_{a=1}^{N_G} (M^{\Omega_B \Omega_{W_a}}_{2})_{;\theta_{G_a}} \Omega_{G_a}\, \Omega_{W_a} \\
&+ \sum_{a=1}^{N_G} (M^{\Omega_B s_a}_{2})[s_a, \theta_{G_a}]\, s_a'' + \sum_{a=1}^{N_G} (M^{\Omega_B s_a}_{2})_{;s_a} (s_a')^2 \\
&+ \sum_{a=1}^{N_G} (M^{\Omega_B s_a}_{2})_{;\theta_{G_a}} \Omega_{G_a}\, s_a' + (M^{\Omega_B q_B}_{2\ c})[s_a, \theta_{G_a};\ a = 1, N_G]\, q_B^{c\prime\prime} \\
&+ \sum_{a=1}^{N_G} (M^{\Omega_B q_B}_{2\ c})_{;s_a} s_a'\, q_B^{c\prime} + \sum_{a=1}^{N_G} (M^{\Omega_B q_B}_{2\ c})_{;\theta_{G_a}} \Omega_{G_a}\, q_B^{c\prime} \\
&+ \sum_{a=1}^{N_G} (M^{\Omega_B q_{G_a}}_{2\ c})[s_a, \theta_{G_a}]\, q_{G_a}^{c\prime\prime} + \sum_{a=1}^{N_G} (M^{\Omega_B q_{G_a}}_{2\ c})_{;s_a} s_a'\, q_{G_a}^{c\prime} \\
&+ \sum_{a=1}^{N_G} (M^{\Omega_B q_{G_a}}_{2\ c})_{;\theta_{G_a}} \Omega_{G_a}\, q_{G_a}^{c\prime}
\end{aligned}
$$

$$+ V_B{}^3 \left((M^{V_B \Omega_B}{}_{1\,c})[s_a, \theta_{G_a}; a = 1, N_G]\, \Omega_B{}^c \right.$$

$$+ \sum_{a=1}^{N_G} (M^{V_B \Omega_{G_a}}{}_{1})[s_a, \theta_{G_a}]\, \Omega_{G_a} + \sum_{a=1}^{N_G} (M^{V_B s_a}{}_{1})[s_a, \theta_{G_a}]\, s_a'$$

$$\left. + (M^{V_B q_B}{}_{1\,c})[s_a, \theta_{G_a}; a = 1, N_G]\, q_B{}^{c\prime} + \sum_{a=1}^{N_G} (M^{V_B q_{G_a}}{}_{1\,c})[s_a, \theta_{G_a}]\, q_{G_a}{}^{c\prime} \right)$$

$$- V_B{}^1 \left((M^{V_B \Omega_B}{}_{3\,c})[s_a, \theta_{G_a}; a = 1, N_G]\, \Omega_B{}^c \right.$$

$$+ \sum_{a=1}^{N_G} (M^{V_B \Omega_{G_a}}{}_{3})[s_a, \theta_{G_a}]\, \Omega_{G_a} + \sum_{a=1}^{N_G} (M^{V_B s_a}{}_{3})[s_a, \theta_{G_a}]\, s_a'$$

$$\left. + (M^{V_B q_B}{}_{3\,c})[s_a, \theta_{G_a}; a = 1, N_G]\, q_B{}^{c\prime} + \sum_{a=1}^{N_G} (M^{V_B q_{G_a}}{}_{3\,c})[s_a, \theta_{G_a}]\, q_{G_a}{}^{c\prime} \right)$$

$$+ \Omega_B{}^3 \left((M^{\Omega_B \Omega_B}{}_{1\,c})[s_a, \theta_{G_a}; a = 1, N_G]\, \Omega_B{}^c \right.$$

$$+ (M^{V_B \Omega_B}{}_{b\,1})[s_a, \theta_{G_a}; a = 1, N_G]\, V_B{}^b$$

$$+ \sum_{a=1}^{N_G} (M^{\Omega_B \Omega_{G_a}}{}_{1})[s_a, \theta_{G_a}]\, \Omega_{G_a} + \sum_{a=1}^{N_G} (M^{\Omega_B \Omega_{W_a}}{}_{1})[s_a, \theta_{G_a}]\, \Omega_{W_a}$$

$$+ \sum_{a=1}^{N_G} (M^{\Omega_B s_a}{}_{1})[s_a, \theta_{G_a}]\, s_a' + (M^{\Omega_B q_B}{}_{1\,c})[s_a, \theta_{G_a}; a = 1, N_G]\, q_B{}^{c\prime}$$

$$\left. + \sum_{a=1}^{N_G} (M^{\Omega_B q_{G_a}}{}_{1\,c})[s_a, \theta_{G_a}]\, q_{G_a}{}' \right)$$

$$- \Omega_B{}^1 \left((M^{\Omega_B \Omega_B}{}_{3\,c})[s_a, \theta_{G_a}; a = 1, N_G]\, \Omega_B{}^c \right.$$

$$+ (M^{v_B \Omega_B}{}_{b\,3})[s_a, \theta_{G_a}; a = 1, N_G]\, v_B{}^b$$

$$+ \sum_{a=1}^{N_G} (M^{\Omega_B \Omega_{G_a}}{}_{3})[s_a, \theta_{G_a}]\, \Omega_{G_a} + \sum_{a=1}^{N_G} (M^{\Omega_B \Omega_{W_a}}{}_{3})[s_a, \theta_{G_a}]\, \Omega_{W_a}$$

$$+ \sum_{a=1}^{N_G} (M^{\Omega_B s_a}{}_{3})[s_a, \theta_{G_a}]\, s_a{}' + (M^{\Omega_B q_B}{}_{3\,c})[s_a, \theta_{G_a}; a = 1, N_G]\, q_B{}^{c\prime}$$

$$\left. + \sum_{a=1}^{N_G} (M^{\Omega_B q_{G_a}}{}_{3\,c})[s_a, \theta_{G_a}]\, q_{G_a}{}' \right)$$

$$= \sum_{a=1}^{N_G} F_{G_a}{}^b \left(e_{2dh}\, \gamma_{B_f}^{g}\, l_{T_a}{}^d\, \delta_{bg}\, \delta^{fh} \right.$$

$$+ e_{2dh}\, \gamma_{B_f}^{g}\, \alpha_{G_{a_e}}{}^d\, (g_{A_a}{}^e + \xi_{A_a}{}^e[s_a])\, \delta_{bg}\, \delta^{fh}$$

$$\left. + e_{fdb}\, \gamma_{B_2}^{f}\, w_{C_a}{}^d + e_{f3b}\, \gamma_{B_2}^{f}\, v_{W_a} \right)$$

$$+ g\, (M^{v_B \Omega_B}{}_{c\,2})[s_a, \theta_{G_a}; a = 1, N_G]\, \gamma_{B_d}^{3}\, \delta^{cd}$$

$$+ F_H{}^b\, e_{2dh}\, \gamma_{B_f}^{g}\, l_H{}^d\, \delta_{bg}\, \delta^{fh} + F_P{}^d\, e_{2cd}\, l_{NP}{}^c + M_P{}^b\, \delta_{2b} + F_T{}^c\, e_{2bc}\, l_T{}^b.$$

The $\Omega_B{}^3$ equation is

$$
\begin{aligned}
&(M^{\Omega_B\Omega_B}_{3\ \ c})[s_a, \theta_{G_a}; a = 1, N_G]\,\Omega_B{}^{c\prime} + \sum_{a=1}^{N_G} (M^{\Omega_B\Omega_B}_{3\ \ c})_{;s_a} \Omega_B{}^c\, s_a{}' \qquad (19)\\
&+ \sum_{a=1}^{N_G} (M^{\Omega_B\Omega_B}_{3\ \ c})_{;\theta_{G_a}} \Omega_B{}^c \Omega_{G_a} + (M^{V_B\Omega_B}_{b\ \ 3})[s_a, \theta_{G_a}; a = 1, N_G]\, V_B{}^{b\prime}\\
&+ \sum_{a=1}^{N_G} (M^{V_B\Omega_B}_{b\ \ 3})_{;s_a} s_a{}'\, V_B{}^{b\prime} + \sum_{a=1}^{N_G} (M^{V_B\Omega_B}_{b\ \ 3})_{;\theta_{G_a}} \Omega_{G_a} V_B{}^{b\prime}\\
&+ \sum_{a=1}^{N_G} (M^{\Omega_B\Omega_{G_a}}_{3})[s_a, \theta_{G_a}]\,\Omega_{G_a}{}' + \sum_{a=1}^{N_G} (M^{\Omega_B\Omega_{G_a}}_{3})_{;s_a} s_a{}' \Omega_{G_a}\\
&+ \sum_{a=1}^{N_G} (M^{\Omega_B\Omega_{G_a}}_{3})_{;\theta_{G_a}} (\Omega_{G_a})^2 + \sum_{a=1}^{N_G} (M^{\Omega_B\Omega_{W_a}}_{3})[s_a, \theta_{G_a}]\,\Omega_{W_a}{}'\\
&+ \sum_{a=1}^{N_G} (M^{\Omega_B\Omega_{W_a}}_{3})_{;s_a} s_a{}' \Omega_{W_a} + \sum_{a=1}^{N_G} (M^{\Omega_B\Omega_{W_a}}_{3})_{;\theta_{G_a}} \Omega_{G_a} \Omega_{W_a}\\
&+ \sum_{a=1}^{N_G} (M^{\Omega_B\, s_a}_{3})[s_a, \theta_{G_a}]\, s_a{}'' + \sum_{a=1}^{N_G} (M^{\Omega_B\, s_a}_{3})_{;s_a} (s_a{}')^2\\
&+ \sum_{a=1}^{N_G} (M^{\Omega_B\, s_a}_{3})_{;\theta_{G_a}} \Omega_{G_a} s_a{}' + (M^{\Omega_B\, q_B}_{3\ \ c})[s_a, \theta_{G_a}; a = 1, N_G]\, q_B{}^{c''}\\
&+ \sum_{a=1}^{N_G} (M^{\Omega_B\, q_B}_{3\ \ c})_{;s_a} s_a{}'\, q_B{}^{c\prime} + \sum_{a=1}^{N_G} (M^{\Omega_B\, q_B}_{3\ \ c})_{;\theta_{G_a}} \Omega_{G_a} q_B{}^{c\prime}\\
&+ \sum_{a=1}^{N_G} (M^{\Omega_B\, q_{G_a}}_{3\ \ c})[s_a, \theta_{G_a}]\, q_{G_a}{}^{c''} + \sum_{a=1}^{N_G} (M^{\Omega_B\, q_{G_a}}_{3\ \ c})_{;s_a} s_a{}'\, q_{G_a}{}^{c\prime}\\
&+ \sum_{a=1}^{N_G} (M^{\Omega_B\, q_{G_a}}_{3\ \ c})_{;\theta_{G_a}} \Omega_{G_a} q_{G_a}{}^{c\prime}
\end{aligned}
$$

$$- V_B^{\,2}\,((M^{V_B \Omega_B}_{1\;c})[s_a,\, \theta_{G_a};\ a = 1,\, N_G]\,\Omega_B^{\,c}$$

$$+ \sum_{a=1}^{N_G} (M^{V_B \Omega_{G_a}}_{1})[s_a,\, \theta_{G_a}]\,\Omega_{G_a} + \sum_{a=1}^{N_G} (M^{V_B s_a}_{1})[s_a,\, \theta_{G_a}]\; s_a{}'$$

$$+ (M^{V_B q_B}_{1\;c})[s_a,\, \theta_{G_a};\ a = 1,\, N_G]\; q_B^{\,c\prime} + \sum_{a=1}^{N_G} (M^{V_B q_{G_a}}_{1\;c})[s_a,\, \theta_{G_a}]\; q_{G_a}^{\,c\prime})$$

$$+ V_B^{\,1}\,((M^{V_B \Omega_B}_{2\;c})[s_a,\, \theta_{G_a};\ a = 1,\, N_G]\,\Omega_B^{\,c}$$

$$+ \sum_{a=1}^{N_G} (M^{V_B \Omega_{G_a}}_{2})[s_a,\, \theta_{G_a}]\,\Omega_{G_a} + \sum_{a=1}^{N_G} (M^{V_B s_a}_{2})[s_a,\, \theta_{G_a}]\; s_a{}'$$

$$+ (M^{V_B q_B}_{2\;c})[s_a,\, \theta_{G_a};\ a = 1,\, N_G]\; q_B^{\,c\prime} + \sum_{a=1}^{N_G} (M^{V_B q_{G_a}}_{2\;c})[s_a,\, \theta_{G_a}]\; q_{G_a}^{\,c\prime})$$

$$-\Omega_B^{\,2}\,((M^{\Omega_B \Omega_B}_{1\;c})[s_a,\, \theta_{G_a};\ a = 1,\, N_G]\,\Omega_B^{\,c}$$

$$+ (M^{V_B \Omega_B}_{b\;1})[s_a,\, \theta_{G_a};\ a = 1,\, N_G]\; V_B^{\,b}$$

$$+ \sum_{a=1}^{N_G} (M^{\Omega_B \Omega_{G_a}}_{1})[s_a,\, \theta_{G_a}]\,\Omega_{G_a} + \sum_{a=1}^{N_G} (M^{\Omega_B \Omega_{W_a}}_{1})[s_a,\, \theta_{G_a}]\,\Omega_{W_a}$$

$$+ \sum_{a=1}^{N_G} (M^{\Omega_B s_a}_{1})[s_a,\, \theta_{G_a}]\; s_a{}' + (M^{\Omega_B q_B}_{1\;c})[s_a,\, \theta_{G_a};\ a = 1,\, N_G]\; q_B^{\,c\prime}$$

$$+ \sum_{a=1}^{N_G} (M^{\Omega_B q_{G_a}}_{1\;c})[s_a,\, \theta_{G_a}]\; q_{G_a}{}')$$

$$+ \Omega_B{}^1 \left((M^{\Omega_B \Omega_B}{}_{2\,c})[s_a, \theta_{G_a}; a = 1, N_G]\, \Omega_B{}^c \right.$$

$$+ (M^{V_B \Omega_B}{}_{b\,2})[s_a, \theta_{G_a}; a = 1. N_G]\, V_B{}^b$$

$$+ \sum_{a=1}^{N_G} (M^{\Omega_B \Omega_{G_a}}{}_{2})[s_a, \theta_{G_a}]\, \Omega_{G_a} + \sum_{a=1}^{N_G} (M^{\Omega_B \Omega_{W_a}}{}_{2})[s_a, \theta_{G_a}]\, \Omega_{W_a}$$

$$+ \sum_{a=1}^{N_G} (M^{\Omega_B s_a}{}_{2})[s_a, \theta_{G_a}]\, s_a{}' + (M^{\Omega_B q_B}{}_{2\,c})[s_a, \theta_{G_a}; a = 1, N_G]\, q_B{}^{c\,\prime}$$

$$\left. + \sum_{a=1}^{N_G} (M^{\Omega_B q_{G_a}}{}_{2\,c})[s_a, \theta_{G_a}]\, q_{G_a}{}' \right)$$

$$= \sum_{a=1}^{N_G} F_{G_a}{}^b \left(e_{3dh}\, \gamma_{B_f}^{\,g}\, l_{T_a}{}^d\, \delta_{bg}\, \delta^{fh} \right.$$

$$+ e_{3dh}\, \gamma_{B_f}^{\,g}\, \alpha_{G_{a_e}}{}^d\, (g_{A_a}{}^e + \xi_{A_a}{}^e[s_a])\, \delta_{bg}\, \delta^{fh}$$

$$\left. + e_{fdb}\, \gamma_{B_3}^{\,f}\, w_{C_a}{}^d + e_{f3b}\, \gamma_{B_3}^{\,f}\, v_{W_a} \right)$$

$$+ g\, (M^{V_B \Omega_B}{}_{c\,3})[s_a, \theta_{G_a}; a = 1, N_G]\, \gamma_{B_d}^{\,3}\, \delta^{cd}$$

$$+ F_H{}^b \left(e_{3dh}\, \gamma_{B_f}^{\,g}\, l_H{}^d\, \delta_{bg}\, \delta^{fh} \right.$$

$$+ (\gamma_{B_3}^{\,g}\, \alpha_{H_3}{}^h\, \gamma_{H_h}^{\,p}\, \delta_{pg})\, e_{sdr}$$

$$\left. \times\, \alpha_{H_3}{}^f\, \gamma_{H_f}^{\,d}\, (a_1\, \gamma_{B_1}^{\,r} + b_1\, \gamma_{B_3}^{\,r})\, \delta_b^{\,s} \right) + F_P{}^d\, e_{3cd}\, l_{NP}{}^c + M_P{}^b\, \delta_{3b}.$$

The $q_B{}^c$ equation is

$$
\begin{aligned}
&(M^{q_B\, q_B}{}_{b\ c})[s_a, \theta_{G_a};\ a = 1, N_G]\ q_B{}^{b''} + (M^{V_B\, q_B}{}_{b\ c})[s_a, \theta_{G_a};\ a = 1, N_G]\ V_B{}^{b'} \\
&+ (M^{\Omega_B\, q_B}{}_{b\ c})[s_a, \theta_{G_a};\ a = 1, N_G]\ \Omega_B{}^{b'} + \sum_{a=1}^{N_G} (M^{\Omega_{G_a} q_B}{}_{c})[s_a, \theta_{G_a}]\ \Omega_{G_a}{}' \\
&+ \sum_{a=1}^{N_G} (M^{s_a q_B}{}_{c})[s_a, \theta_{G_a}]\ s_a{}'' + \sum_{a=1}^{N_G} (M^{q_B\, q_{G_a}}{}_{c\ b})[s_a, \theta_{G_a}]\ q_{G_a}{}^{b''} \\
&+ \sum_{a=1}^{N_G} (M^{q_B\, q_B}{}_{b\ c})_{;s_a}\ q_B{}^{b'}\ s_a{}' + \sum_{a=1}^{N_G} (M^{V_B\, q_B}{}_{b\ c})_{;s_a}\ V_B{}^{b}\ s_a{}' \\
&+ \sum_{a=1}^{N_G} (M^{\Omega_B\, q_B}{}_{b\ c})_{;s_a}\ \Omega_B{}^{b}\ s_a{}' + \sum_{a=1}^{N_G} (M^{\Omega_{G_a} q_B}{}_{c})_{;s_a}\ \Omega_{G_a}\ s_a{}' \\
&+ \sum_{a=1}^{N_G} (M^{s_a q_B}{}_{c})_{;s_a}\ (s_a{}')^2 + \sum_{a=1}^{N_G} (M^{q_B\, q_{G_a}}{}_{c\ b})_{;s_a}\ q_{G_a}{}^{b'}\ s_a{}' \\
&+ \sum_{a=1}^{N_G} (M^{q_B\, q_B}{}_{b\ c})_{;\theta_{G_a}}\ q_B{}^{b'}\ \Omega_{G_a} + \sum_{a=1}^{N_G} (M^{V_B\, q_B}{}_{b\ c})_{;\theta_{G_a}}\ V_B{}^{b}\ \Omega_{G_a} \\
&+ \sum_{a=1}^{N_G} (M^{\Omega_B\, q_B}{}_{b\ c})_{;\theta_{G_a}}\ \Omega_B{}^{b}\ \Omega_{G_a} + \sum_{a=1}^{N_G} (M^{\Omega_{G_a} q_B}{}_{c})_{;\theta_{G_a}}\ (\Omega_{G_a})^2 \\
&+ \sum_{a=1}^{N_G} (M^{s_a q_B}{}_{c})_{;\theta_{G_a}}\ s_a{}'\ \Omega_{G_a} + \sum_{a=1}^{N_G} (M^{q_B\, q_{G_a}}{}_{c\ b})_{;\theta_{G_a}}\ q_{G_a}{}^{b'}\ \Omega_{G_a}
\end{aligned}
\qquad (20)
$$

$$= \sum_{a=1}^{N_G} F_{G_a}{}^{b} \gamma_{G_{a_h}}{}^{e} \gamma_{B_e}{}^{g} \Phi_{BG_{A_{a_c}}}{}^{h}[s_a, \theta_{G_a}] \delta_{bg}$$

$$+ F_H{}^{b} (\gamma_{B_d}{}^{e} \Phi_{B_{H_c}}{}^{d} \delta_{be}$$

$$+ (\Psi_{B_{H_c}}{}^{e} \gamma_{B_e}{}^{g} \alpha_{H_3}{}^{h} \gamma_{H_h}{}^{p} \delta_{pg}) e_{sdr}$$

$$\times \alpha_{H_3}{}^{f} \gamma_{H_f}{}^{d} (a \gamma_{B_1}{}^{r} + b \gamma_{B_3}{}^{r}) \delta_b^s)$$

$$+ g (M^{V_B}{}_{b}{}^{q_B}{}_{c})[s_d, \theta_{G_d}; d = 1, N_G] \gamma_{B_e}{}^{3} \delta^{be}$$

$$+ Q_A{}^{b} \delta_{bc}$$

$$- K_{B_c} q_B{}^{d} \delta_{cd} - C_{B_c} q_B{}^{d\prime} \delta_{cd}.$$

The s_a equation is

$$
\begin{aligned}
&(M^{s_a s_a})[s_a]\, s_a'' + (M^{V_B s_a}{}_b)[s_a, \theta_{G_a}]\, V_B{}^{b\prime} \qquad (21)\\
&+ (M^{\Omega_B s_a}{}_b)[s_a, \theta_{G_a}]\, \Omega_B{}^{b\prime} + (M^{\Omega_{G_a} s_a})[s_a]\, \Omega_{G_a}{}'\\
&+ (M^{s_a q_B}{}_b)[s_a, \theta_{G_a}]\, q_B{}^{b\prime\prime} + (M^{s_a q_{G_a}}{}_b)[s_a, \theta_{G_a}]\, q_{G_a}{}^{b\prime\prime}\\
&+ 1/2\, (M^{s_a s_a})_{;s_a} (s_a{}')^2 + (M^{V_B s_a}{}_b)_{;\theta_{G_a}} V_B{}^b\, \Omega_{G_a}\\
&+ (M^{\Omega_B s_a}{}_b)_{;\theta_{G_a}} \Omega_B{}^b\, \Omega_{G_a} + (M^{s_a q_B}{}_b)_{;\theta_{G_a}} q_B{}^{b\prime}\, \Omega_{G_a} + (M^{s_a q_{G_a}}{}_b)_{;\theta_{G_a}} q_{G_a}{}^{b\prime}\, \Omega_{G_a}\\
&- 1/2\, (M^{\Omega_B \Omega_B}{}_{b\,c})_{;s_a} \Omega_B{}^b\, \Omega_B{}^c - 1/2\, (M^{\Omega_{G_a} \Omega_{G_a}})_{;s_a} (\Omega_{G_a})^2\\
&- 1/2\, (M^{q_B q_B}{}_{b\,c})_{;s_a} q_B{}^{b\prime}\, q_B{}^{c\prime} - 1/2\, (M^{q_{G_a} q_{G_a}}{}_{b\,c})_{;s_a} q_{G_a}{}^{b\prime}\, q_{G_a}{}^{c\prime}\\
&- (M^{V_B \Omega_B}{}_{b\,c})_{;s_a} V_B{}^b\, \Omega_B{}^c - (M^{V_B \Omega_{G_a}}{}_b)_{;s_a} V_B{}^b\, \Omega_{G_a}\\
&- (M^{V_B q_B}{}_{b\,c})_{;s_a} V_B{}^b\, q_B{}^{c\prime} - (M^{V_B q_{G_a}}{}_b)_{;s_a} V_B{}^b\, q_{G_a}{}^{c\prime}\\
&- (M^{\Omega_B \Omega_{G_a}}{}_b)_{;s_a} \Omega_B{}^b\, \Omega_{G_a} - (M^{\Omega_B q_B}{}_{b\,c})_{;s_a} \Omega_B{}^b\, q_B{}^{c\prime}\\
&- (M^{\Omega_B q_{G_a}}{}_{b\,c})_{;s_a} \Omega_B{}^b\, q_{G_a}{}^{c\prime} - (M^{\Omega_{G_a} q_B}{}_b)_{;s_a} \Omega_{G_a}\, q_B{}^{b\prime}
\end{aligned}
$$

$$- (M^{\Omega_{G_a} q_{G_b}})_{;s_a} \Omega_{G_a} q_{G_a}{}^{b\prime} - (M^{q_{B_b} q_{G_c}})_{;s_a} q_B{}^{b\prime} q_{G_a}{}^{c\prime}$$

$$= F_{S_a} + F_{G_a}{}^{b} (\alpha_{G_{a_c}}{}^{e} \gamma_{B_e}{}^{g} \xi_{A_a}{}^{c\prime}[s_a] \delta_{bg})$$

$$+ g (M^{V_B s_a}{}_{c})[s_a, \theta_{G_a}] \gamma_{B_d}^{3} \delta^{cd}.$$

The Ω_{W_a} equation is

$$(M^{\Omega_{W_a} \Omega_{W_a}}) \Omega_{W_a}{}' + (M^{\Omega_{B_b} \Omega_{W_a}})[s_a, \theta_{G_a}] \Omega_B{}^{b\prime} \tag{22}$$

$$+ (M^{\Omega_{B_b} \Omega_{W_a}})_{;s_a} s_a{}' \Omega_B{}^{b} + (M^{\Omega_{B_b} \Omega_{W_a}})_{;\theta_{G_a}} \Omega_{G_a} \Omega_B{}^{b}$$

$$+ (M^{\Omega_{G_a} \Omega_{W_a}})[s_a] \Omega_{G_a}{}'$$

$$= F_{G_a}{}^{b} (e_{cdb} \beta_{G_{a_2}}{}^{p} \alpha_{G_{a_p}}{}^{e} \gamma_{B_e}{}^{c} w_{C_a}{}^{d}$$

$$+ e_{c3b} \beta_{G_{a_2}}{}^{p} \alpha_{G_{a_p}}{}^{e} \gamma_{B_e}{}^{c} v_{W_a}).$$

The $q_{G_a}{}^{c}$ equation is

$$
\begin{aligned}
&(M^{q_{G_a}}{}_{b}{}^{q_{G_a}}{}_{c})[s_a, \theta_{G_a}]\, q_{G_a}{}^{b''} + (M^{V_B}{}_{b}{}^{q_{G_a}}{}_{c})[s_a, \theta_{G_a}]\, V_B{}^{b'} \\
&+ (M^{\Omega_B}{}_{b}{}^{q_{G_a}}{}_{c})[s_a, \theta_{G_a}]\, \Omega_B{}^{b'} + (M^{s_a q_{G_a}}{}_{c})[s_a, \theta_{G_a}]\, s_a'' \\
&+ (M^{q_B}{}_{b}{}^{q_{G_a}}{}_{c})[s_a, \theta_{G_a}]\, q_B{}^{b''} + (M^{q_{G_a}}{}_{b}{}^{q_{G_a}}{}_{c})_{;s_a}\, q_{G_a}{}^{b'}\, s_a' \\
&+ (M^{V_B}{}_{b}{}^{q_{G_a}}{}_{c})_{;s_a}\, V_B{}^{b}\, s_a' + (M^{\Omega_B}{}_{b}{}^{q_{G_a}}{}_{c})_{;s_a}\, \Omega_B{}^{b}\, s_a' \\
&+ (M^{s_a q_{G_a}}{}_{c})_{;s_a}\, (s_a')^2 + (M^{q_B}{}_{b}{}^{q_{G_a}}{}_{c})_{;s_a}\, q_B{}^{b'}\, s_a' + (M^{q_{G_a}}{}_{b}{}^{q_{G_a}}{}_{c})_{;\theta_{G_a}}\, q_{G_a}{}^{b'}\, \Omega_{G_a} \\
&+ (M^{V_B}{}_{b}{}^{q_{G_a}}{}_{c})_{;\theta_{G_a}}\, V_B{}^{b}\, \Omega_{G_a} + (M^{\Omega_B}{}_{b}{}^{q_{G_a}}{}_{c})_{;\theta_{G_a}}\, \Omega_B{}^{b}\, \Omega_{G_a} \\
&+ (M^{s_a q_{G_a}}{}_{c})_{;\theta_{G_a}}\, s_a'\, \Omega_{G_a} + (M^{q_B}{}_{b}{}^{q_{G_a}}{}_{c})_{;\theta_{G_a}}\, q_B{}^{b'}\, \Omega_{G_a} \\
&= F_{G_a}{}^{b}\, (\Omega_{G_{a_h}}{}^{e}\, \gamma_{B_e}{}^{g}\, \Phi_{G_{A_{a_c}}}{}^{h}[s_a, \theta_{G_a}]\, \delta_{bg}) + g\, (M^{V_B}{}_{b}{}^{q_{G_a}}{}_{c})[s_a, \theta_{G_a}]\, \gamma_{B_d}{}^{3}\, \delta^{bd} \\
&- K_{G_{a_c}}[s_a, \theta_{G_a}]\, q_{G_a}{}^{b}\, \delta_{bc} - C_{G_{a_c}}[s_a, \theta_{G_a}]\, q_{G_a}{}^{b'}\, \delta_{bc}.
\end{aligned}
\tag{23}
$$

The Ω_{G_a} equation is

$$
\begin{aligned}
&(M^{\Omega_{G_a}\Omega_{G_a}})[s_a]\,\Omega_{G_a}{}' + (M^{V_B\,\Omega_{G_a}}{}_{b})[s_a,\,\theta_{G_a}]\,V_B{}^{b\prime} \qquad (24)\\
&+ (M^{\Omega_B\,\Omega_{G_a}}{}_{b})[s_a,\,\theta_{G_a}]\,\Omega_B{}^{b\prime} + (M^{\Omega_{G_a}\Omega_{W_a}})[s_a]\,\Omega_{W_a}{}'\\
&+ (M^{\Omega_{G_a}s_a})[s_a]\,s_a{}'' + (M^{\Omega_{G_a}q_B}{}_{b})[s_a,\,\theta_{G_a}]\,q_B{}^{b\prime\prime}\\
&+ (M^{\Omega_{G_a}q_{G_a}}{}_{b})[s_a,\,\theta_{G_a}]\,q_{G_a}{}^{b\prime\prime} + (M^{\Omega_{G_a}\Omega_{G_a}})_{;s_a}\,\Omega_{G_a}\,s_a{}'\\
&+ (M^{V_B\,\Omega_{G_a}}{}_{b})_{;s_a}\,s_a{}'\,V_B{}^{b} + (M^{\Omega_B\,\Omega_{G_a}}{}_{b})_{;s_a}\,\Omega_B{}^{b}\,s_a{}' + (M^{\Omega_{G_a}\Omega_{W_a}})_{;s_a}\,\Omega_{W_a}\,s_a{}'\\
&+ (M^{\Omega_{G_a}s_a})_{;s_a}\,(s_a{}')^2 + (M^{\Omega_{G_a}q_B}{}_{b})_{;s_a}\,s_a{}'\,q_B{}^{b\prime}\\
&+ (M^{\Omega_{G_a}q_{G_a}})_{;s_a}\,s_a{}'\,q_{G_a}{}^{b\prime} - 1/2\,(M^{\Omega_B\,\Omega_B}{}_{b\,c})_{;\theta_{G_a}}\,\Omega_B{}^{b}\,\Omega_B{}^{c}\\
&- 1/2\,(M^{q_B\,q_B}{}_{b\,c})_{;\theta_{G_a}}\,q_B{}^{b\prime}\,q_B{}^{c\prime} - 1/2\,(M^{q_{G_a}\,q_{G_a}}{}_{b\,c})_{;\theta_{G_a}}\,q_{G_a}{}^{b\prime}\,q_{G_a}{}^{c\prime}\\
&- (M^{V_B\,\Omega_B}{}_{b\,c})_{;\theta_{G_a}}\,V_B{}^{b}\,\Omega_B{}^{c\prime} - (M^{V_B\,s_a}{}_{b})_{;\theta_{G_a}}\,V_B{}^{b}\,s_a{}'\\
&- (M^{V_B\,q_B}{}_{b\,c})_{;\theta_{G_a}}\,V_B{}^{b}\,q_B{}^{c\prime} - (M^{V_B\,q_{G_a}}{}_{b\,c})_{;\theta_{G_a}}\,V_B{}^{b}\,q_{G_a}{}^{c\prime}\\
&- (M^{\Omega_B\,s_a}{}_{b})_{;\theta_{G_a}}\,\Omega_B{}^{b}\,s_a{}' - (M^{\Omega_B\,q_B}{}_{b\,c})_{;\theta_{G_a}}\,\Omega_B{}^{b}\,q_B{}^{c\prime}\\
&- (M^{\Omega_B\,q_{G_a}}{}_{b\,c})_{;\theta_{G_a}}\,\Omega_B{}^{b}\,q_{G_a}{}^{c\prime} - (M^{s_a q_B}{}_{b})_{;\theta_{G_a}}\,s_a{}'\,q_B{}^{b\prime}\\
&- (M^{s_a q_{G_a}}{}_{b})_{;\theta_{G_a}}\,q_{G_a}{}'\,s_a{}' - (M^{q_B\,q_{G_a}}{}_{b\,c})_{;\theta_{G_a}}\,q_B{}^{b\prime}\,q_{G_a}{}^{c\prime}
\end{aligned}
$$

$$= F_{G_a}{}^{b} \left(\alpha_{G_{a_d}}{}^{e} \gamma_{B_e}^{g} e_{3hc} \left(g_{A_a}{}^{h} + \xi_{A_a}{}^{h}[s_a] \right) \delta^{cd} \delta_{bg} \right.$$

$$+ e_{deb} \alpha_{G_{a_3}}{}^{c} \gamma_{B_c}^{d} w_{C_a}{}^{e} + e_{d3b} \alpha_{G_{a_3}}{}^{c} \gamma_{B_c}^{d} v_{W_a} \Big)$$

$$+ g \left(M_{c}^{v_B \Omega_{G_a}} \right)[s_a, \theta_{G_a}] \gamma_{B_d}^{3} \delta^{cd} - M_C[\theta_{G_a}, \Omega_{G_a}].$$

Initial Conditions

The aircraft is assumed to be oriented at t = 0 such that one point of the aircraft has just made contact with the ground surface. The candidate points for contact are the landing gear tires and the arresting hook. There are some cases when there are multiple points of simultaneous contact at the time t = 0.

The transformation $\gamma_{B_c}^{\ b}(0)$; b = 1, 3 and c = 1, 3 may be established from the Euler angles $\psi(0)$, $\theta(0)$ and $\phi(0)$ which are the initial yaw, pitch and roll angles of the airframe. Appendix G defines the elements of this transformation.

The rigid body translational velocity of the airframe reference point N at t = 0 is

$$\bar{r}'(0) = V_B^{\ c}(0)\ \bar{i}_{B_c}(0) = \zeta^{b\prime}(0)\ \bar{I}_b = d'(0)\ \bar{I}_1 + s'(0)\ \bar{I}_2 + v'(0)\ \bar{I}_3.$$

Normally, the horizontal velocity, the vertical velocity or sinking velocity and the lateral velocity of the point N with respect to the ground surface (i.e. $\zeta^{1\prime}(0)\ \bar{I}_1$, $\zeta^{2\prime}(0)\ \bar{I}_2$ and $\zeta^{3\prime}(0)\ \bar{I}_3$) are specified at t = 0. Therefore, the body axis components of the initial velocity are

$$V_B^{\ e}(0) = \zeta^{b\prime}(0)\ \gamma_{B_d}^{\ f}(0)\ \delta_{bf}\ \delta^{de}.$$

Also, when the initial angular velocity is not zero, the components of

the initial angular velocity may be expressed in terms of the initial Euler angle rates as follows:

$$\Omega_B{}^1(0) = \phi'(0) - \sin(\theta(0))\ \psi'(0).$$

$$\Omega_B{}^2(0) = \cos(\phi(0))\ \theta'(0) + \cos(\theta(0))\ \sin(\phi(0))\ \psi'(0).$$

$$\Omega_B{}^3(0) = -\sin(\phi(0))\ \theta'(0) + \cos(\theta(0))\ \cos(\phi(0))\ \psi'(0).$$

The following bth airframe modal displacement for b in [1, N_{BE}] at t = 0 is derived from equation (20). It is assumed that there may be cases where the initial castor angle is not zero. It is further assumed that the functions that are dependent on the stroke of the landing gears can be evaluated with the initial stroke equal to zero then

$$q_B{}^b(0) = \frac{1}{K_{B_b}} \left(g\ (M^{V_B\ q_B}{}_{c\ \ b})(0,\ \theta_{G_d}(0);\ d = 1,\ N_G)\ \gamma_{B_e}{}^3\ \delta^{ce} + Q_A{}^b\right).$$

The initial position of the point N with respect to the ground reference point Q is defined as follows:

$$\zeta^1(0) = d(0) = \bar{r}(0) \cdot \bar{I}_1.$$

$$\zeta^2(0) = s(0) = \bar{r}(0) \cdot \bar{I}_2.$$

$$\zeta^3(0) = v(0) = \bar{r}(0) \cdot \bar{I}_3.$$

The number d(0) is normally zero, the number s(0) is used to orient the point N laterally with respect to the point S (see Appendix F) and the

number v(0) may be derived from equation (12) for the case where the a gear tire has just contacted the ground surface. It is found from equation (12) that

$$v(0) = -\, l_{T_a}{}^{b}\, \gamma_{B_b}{}^{3}(0) - \Phi_{BG_{A_{a_b}}}{}^{c}(0,\ \theta_{G_a}(0))\, \gamma_{G_{a_c}}{}^{d}\, \gamma_{B_d}{}^{3}(0)\, q_B{}^{b}(0)$$

$$-\, g_{A_a}{}^{b}\, \alpha_{G_{a_b}}{}^{c}(0)\, \gamma_{B_c}{}^{3}(0)$$

$$-\, \Phi_{G_{A_{a_b}}}{}^{c}(0,\ \theta_{G_a}(0))\, \alpha_{G_{a_c}}{}^{d}(0)\, \gamma_{B_d}{}^{3}(0)\, q_{a}{}^{b}(0) - w_{C_a}{}^{3}(0) - v_{W_{a_0}}.$$

The equation for v(0) may be obtained as follows for the case where at t = 0 the arresting hook has just made contact with the ground surface:

$$v(0) = -\, l_H{}^{b}\, \gamma_{B_b}{}^{3}(0) - (-\, l_{HV} \sin(\phi_{H_0})\, \gamma_{B_1}{}^{3}(0) + l_{HV} \cos(\phi_{H_0})\, \gamma_{B_3}{}^{3}(0)).$$

The a gear stroke at t = 0 is determined from the following equation which is derived from equation (21) and the equation for the number $F_{S_a}(0)$ from Appendix B:

$$s_a(0) = \frac{1}{K_{BG_a}} (-\, p_{A_a}(s_a(0))\, A_{A_a}$$

$$+\, 1/2\, (M^{\Omega_B \Omega_B}{}_{b\ c})_{;s_a}(0,\ \theta_{G_d}(0);\ d = 1,\ N_G)\, \Omega_B{}^{b}(0)\, \Omega_B{}^{c}(0)$$

$$+\, (M^{V_B \Omega_B}{}_{b\ c})_{;s_a}(0,\ \theta_{G_d}(0);\ d = 1,\ N_G)\, V_B{}^{b}(0)\, \Omega_B{}^{c}(0)$$

$$+\, g\, (M^{V_B\, s_a}{}_{c})(0,\ \theta_{G_a}(0))\, \gamma_{B_d}{}^{3}(0)\, \delta^{cd}).$$

The hydraulic fluid expansion in the a gear fluid chamber and the a gear snubber chamber may be obtained at t = 0 from Appendix B as follows:

$$\beta_a(0) = -\frac{p_{A_a}(s_a(0))\, V_{H_a}(0)}{K_H}.$$

$$\beta_{S_a}(0) = -\frac{p_{A_a}(s_a(0))\, V_{H_{S_a}}(0)}{K_H}.$$

The following a gear bth modal displacement for b in $[1, N_{GE_a}]$ at t = 0 is derived from equation (23):

$$q_{G_a}{}^{b}(0) = \frac{1}{K_{G_{a_b}}} \left(g\left(M_c^{V_B\, q_{G_a b}}\right)(0,\ \theta_{G_a}(0))\ \gamma_{B_d}{}^{3}(0)\ \delta^{cd}\right).$$

Summary of Vector Function Relations

$$\bar{i}_{B_c} = \gamma_{B_c}{}^{b}\,\bar{I}_b$$

$$\bar{i}_{G_{a_c}} = \gamma_{G_{a_c}}{}^{b}\,\bar{i}_{B_b}$$

$$\bar{j}_{G_{a_b}} = \hat{\alpha}_{G_{a_g}}{}^{e}\,\bar{i}_{G_{a_c}}\,\delta_{be}\,\delta^{cg}$$

$$\bar{j}_{G_{a_b}} = \alpha_{G_{a_b}}{}^{d}\,\bar{i}_{B_d}$$

$$\bar{k}_{G_{a_c}} = \beta_{G_{a_c}}{}^{b}\,\bar{j}_{G_{a_b}}$$

$$= B_{G_{a_c}}{}^{b}[s_a]\,\bar{j}_{G_{a_b}}$$

$$\bar{I}_{W_{a_b}} = \gamma_{W_{a_g}}{}^{e}\,\bar{I}_c\,\delta_{be}\,\delta^{cg}$$

$$\bar{I}_{H_c} = \gamma_{H_c}{}^{b}\,\bar{I}_b$$

$$\bar{i}_{H_d} = \alpha_{H_d}{}^{c}\,\bar{I}_{H_c}$$

$$\bar{r}' = V_B{}^{b}\,\bar{i}_{B_b}$$

$$\bar{\Omega}_B = \Omega_B{}^{b}\,\bar{i}_{B_b}$$

$$\bar{\Omega}_{G_a} = \Omega_{G_a}\,\bar{j}_{G_a}$$

$$\bar{\Omega}_{W_a} = \Omega_{W_a} \bar{k}_{G_{a_2}}$$

$$\bar{L}_B = L_B{}^b \bar{i}_{B_b}$$

$$\bar{l}_{T_a} = l_{T_a}{}^b \bar{i}_{E_b}$$

$$\bar{l}_H = l_H{}^b \bar{i}_{B_b}$$

$$\bar{l}_T = l_T{}^b \bar{i}_{B_b}$$

$$\bar{G}_a = G_a{}^b \bar{j}_{G_{a_b}}$$

$$\bar{\Xi}_a = \Xi_a{}^b \bar{j}_{G_{ab}}$$

$$\bar{\xi}_{A_a} = \xi_{A_a}{}^b \bar{j}_{G_{a_b}}$$

$$\bar{W}_a = W_a{}^b \bar{k}_{G_{a_b}}$$

$$\bar{w}_{C_a} = w_{C_a}{}^b \bar{I}_b$$

$$\bar{v}_{W_a} = v_{W_a} \bar{I}_3$$

$$\bar{\Phi}_{B_b} = \Phi_{B_b}{}^c \bar{i}_{B_c}$$

$$\bar{\Phi}_{BG_{a_b}} = \Phi_{BG_{a_b}}{}^c \bar{I}_{G_{a_c}}$$

$$\bar{\Phi}_{BG_{A_{a_b}}} = \Phi_{BG_{A_{a_b}}}{}^{c}\,\bar{i}_{G_{a_c}}$$

$$\bar{\Phi}_{B_{H_b}} = \Phi_{B_{H_b}}{}^{c}\,\bar{i}_{B_c}$$

$$\bar{\Psi}_{B_{H_b}} = \Psi_{B_{H_b}}{}^{c}\,\bar{i}_{B_c}$$

$$\bar{\Phi}_{G_{a_b}} = \Phi_{G_{a_b}}{}^{c}\,\bar{j}_{G_{a_c}}$$

$$\bar{\Phi}_{G_{A_{a_b}}} = \Phi_{G_{A_{a_b}}}{}^{c}\,\bar{j}_{G_{a_c}}$$

$$\bar{F}_{H} = F_{H}{}^{b}\,\bar{I}_{b}$$

$$\bar{F}_{G_a} = F_{G_a}{}^{b}\,\bar{I}_{b}$$

$$\bar{F}_{P} = F_{P}{}^{b}\,\bar{i}_{B_b}$$

$$\bar{M}_{P} = M_{P}{}^{b}\,\bar{i}_{B_b}$$

$$\bar{F}_{T} = F_{T}{}^{b}\,\bar{i}_{B_b}$$

SIMPLIFICATION OF EQUATIONS

The equations of motion (equations (14) through (24)) in combination with the equations developed in the Appendices provide considerable flexibility for calculating the loads associated with landing impact. A minimum of assumptions were made during the course of the development of these equations to help ensure this flexibility. In many cases, however, the equations may be simplified by virtue of the gear design. For example, for the case where the nose gear power steering unit is engaged and is able to maintain the castor angle at zero, then considerable simplification may be realized.

There are several simplifications that may be made to the equations of motion if the gears are not articulated (i.e. there is a one to one correlation of the shock strut stroke and axle motion). These simplifications are readily determined from an examination of the equations of motion.

There are other simplifications that may be realized. For example, it is usually permissible to assume that the airframe vibration modal effective masses are independent of the shock strut stroke. Therefore, when this approximation is used in conjunction with airframe normal modes that are orthogonal to the rigid body motion, the complexity of the equations of motion is reduced.

Further, it may be possible to delete other terms by computing the mass

terms in Appendix D and estimating the contribution of the terms involving these masses with approximations of the multiplying functions which are found in the equations of motion.

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APPENDIX A

MATHEMATICAL NOTATION

In the text of the report there is frequent use of the terms simple graph and simple surface. This is done to emphasize their functional and geometrical significance.

The statement that "f is a simple graph" means that f is a point set no two members of which have the same abscissa. The number $f(x)$ is the ordinate of that point of the simple graph f whose abscissa is x.

The statement that "T is a transformation" means that T is a collection of one or more ordered pairs no two of which have the same first term.

The statement that "F is a simple surface" means that F is a transformation from a point set to a number set. If the point $P = (x, y)$ and the number z constitute a point of F then $z = F(x, y)$.

The use of the bracket product notation has been used extensively in this report. The following definition of a bracket product may be used to illustrate the essential features of this important functional description.

The statement that "the simple graph $F[g_1, g_2]$ is the bracket product of F of g_1 and g_2 (F is a simple surface and each of g_1 and g_2 is a simple graph)" means that there is a number x such that $(g_1(x), g_2(x))$ is in the x,y projection of F and if x is such a number,

$F[g_1, g_2](x) = F(g_1(x), g_2(x))$.

An extremely useful concept in the handling of functions and their derivatives is the identity graph. This graph contains the point (0,0) and has slope equal to one. Consequently the statement that I is the identity graph means that if x is a number then $I(x) = x$. The identity graph I appears in the text with a subscript to provide clarity when used in combination with simple surfaces.

With the help of the identity graph the definition of the derivative of a simple surface may be stated as follows: The statement that F_1' is the 1-derivative of the simple surface F means that F_1' is the simple surface to which the point ((x, y), z) belongs only if

$z = \{F[I, y]\}'(x)$.

In addition the statement that F_2' is the 2-derivative of the simple surface F means that F_2' is the simple surface to which ((x, y), z) belongs only if

$z = \{F[x, I]\}'(y)$.

These definitions lead to the notation used in the text of the report for partial derivatives. To illustrate this suppose that each of f, g_1, and g_2 is a simple graph and F is a simple surface such that

$f = F[g_1, g_2]$.

The statement that $F_{;g_1}$ is the partial derivative of F with respect to g_1 means that

$F_{;g_1} = F_1'[g_1, g_2]$.

Further, the statement that $F_{;g_2}$ is the partial derivative of F with respect to g_2 means that

$F_{;g_2} = F_2'[g_1, g_2]$.

The identity graph is also useful in the integration process. In the general case f is a simple graph bounded on the interval [a, b] and the simple graph g is nondecreasing on [a,b] then f is g integrable on [a,b] (Reference (3)). This is denoted by

$$\int_a^b f \, dg.$$

In the special case where g = I (the identity graph) this integral (i.e. the integral of f with respect to I) is the well known Reimann integral.

Vectors are denoted by a bar over the symbol. When the vector is expressible in terms of a unit vector set, for example, $\bar{I}_a$; a = 1, 3 such that if $\bar{F}$ is such a vector and

$\bar{F} = F^a \bar{I}_a$ (sum on a is implied),

then each of F^a; a = 1, 3 is referred to as a component of the vector $\bar{F}$.

Frequent use is made of the "e system" for representing vector operations. This system is discussed in detail in Reference (4). The e system may be defined in the following manner:

If each of r, s and t is an integer in the interval [1, 3] then e_{rst} is -1, 0 or +1 according to the formula

$e_{rst} = 1/2 \, (r - s) \, (s - t) \, (t - r)$,

and

$e^{rst} = e_{rst}$.

It is seen that e_{rst} is skew-symmetric in each pair of its indices.

The e system is used in the main body of the report to express vector cross products in terms of vector components. For example, if

$$\bar{A} = A^s \, \bar{i}_s$$

and

$$\bar{B} = B^t \, \bar{i}_t,$$

then

$$\bar{A} \times \bar{B} \, . \, \bar{i}_r = e_{rst} \, A^s \, B^t.$$

The Kronecker deltas used in the text are defined as follows:

$$\delta^{abc}_{rst} = e_{rst} \, e^{abc}.$$

$$\delta^{ab}_{rs} = \delta^{abc}_{rsc}.$$

$$\delta^{a}_{r} = 1/2 \, \delta^{ab}_{rb}.$$

Also,

$\delta^{ab} = \delta_{ab} = 1$ if a is equal to b and

$\delta^{ab} = \delta_{ab} = 0$ if a is unequal to b.

APPENDIX B

DERIVATION OF THE SHOCK STRUT FORCE EQUATIONS

The following shock strut characteristics are assumed for the derivation of the equations for the shock strut pressure and force functions in terms of the quasi-coordinate velocity component functions of the airframe and the generalized coordinate displacement functions for the landing gear:

(1) The shock strut contains a hydraulic fluid and a gas.

(2) During shock strut compression energy is stored in the gas.

(3) During shock strut compression and extension energy is dissipated in the hydraulic fluid and the coulomb friction forces from the shock strut bearings.

(4) The shock strut is equipped with an orifice for control of the hydraulic fluid flow during compression and the area of the orifice is assumed to be dependent only on the stroke of the shock strut.

(5) The shock strut is equipped with a snubbing chamber (see Figure B-1) for energy dissipation during extension of the shock strut and the orifice area of the snubbing chamber is controlled by a pressure relief valve.

(6) The shock strut has a single stage air chamber.

(7) The hydraulic fluid is compressible and the bulk modulus of the hydraulic fluid is assumed to be a number.

To provide some additional flexibility for modeling potential shock strut configurations, two shock strut geometries that have the

characteristics described above will be included in the derivation.

The terms used in the shock strut force derivation are defined below. Figures B-1 and B-3 may be referred to for illustration of the geometrical quantities. The required numbers are defined as follows:

p_{AMB} is the ambient pressure.

A_{H_a} is the hydraulic area of the a gear fluid chamber.

A_{A_a} is the net area exposed to the gas pressure in the a gear.

A_{O_a} is the area of the hole in the main orifice plate in the a gear.

$A_{H_{S_a}}$ is the hydraulic area of the gear snubbing chamber in the a gear.

K_H is the bulk modulus of the hydraulic fluid.

ρ_H is the mass density of the hydraulic fluid.

$V_{A_{0_a}}$ is the gas volume in the a gear gas chamber with the shock strut fully extended.

$V_{H_{0_a}}$ is the volume in the a gear fluid chamber with the with the shock strut fully extended.

K_{BG_a} is the spring rate of the a gear piston-cylinder interface when the strut is fully extended.

C_{BG_a} is the damping rate (assumed viscous) of the a gear piston-cylinder interface when the strut is fully extended.

V_{FL_a} is the fluid volume in the a gear shock strut.

C_{D_a} is the main orifice coefficient for the a gear.

$C_{D_{S_a}}$ is the snubbing chamber orifice coefficient for the a gear.

M_{W_a} is the mass of the rotating parts on the a gear axle.

s_{C_a} is the stroke of the a gear when fully compressed.

μ_{BU} is the upper shock strut bearing friction coefficient.

μ_{BL} is the lower shock strut bearing friction coefficient.

The vector functions needed to define the shock strut force are defined as follows:

$\bar{i}_{S_{a_3}}$ is the unit vector function such that if $t \geq 0$, $\bar{i}_{S_{a_3}}(t)$ is the unit vector which at the time t is parallel to the shock strut axis of symmetry and directed downward with respect to the piston as shown in Figure

B-2.

$\bar{F}_{S_a}$ is the vector function such that if $t \geq 0$, $\bar{F}_{S_a}(t)$ is the shock strut force on the a gear piston at the time t.

F_{S_a} is the simple graph such that if $t \geq 0$,

$\bar{F}_{S_a}(t) = F_{S_a}(t)\, \bar{i}_{S_{a_3}}(t)$ at the time t.

$\bar{v}_{H_a}$ is the vector function such that if $t \geq 0$, $\bar{v}_{H_a}(t)$ is the velocity of the hydraulic fluid passing through the a gear orifice at the time t.

v_{H_a} is the simple graph such that if $t \geq 0$,

$\bar{v}_{H_a}(t) = -\, v_{H_a}(t)\, \bar{i}_{S_{a_3}}(t)$ at the time t.

$\bar{v}_{H_{S_a}}$ is the vector function such that if $t \geq 0$, $\bar{v}_{H_{S_a}}(t)$ is the velocity of the hydraulic fluid passing through the a gear snubbing orifice at the time t.

$v_{H_{S_a}}$ is the simple graph such that if $t \geq 0$,

$\bar{v}_{H_{S_a}}(t) = -\, v_{H_{S_a}}(t)\, \bar{i}_{S_{a_3}}(t)$ at the time t.

$\bar{F}_{F_a}$ is the vector function such that if $t \geq 0$, $\bar{F}_{F_a}(t)$ is the shock strut friction force on the a gear piston at the time t.

F_{F_a} is the simple graph such that if $t \geq 0$

$\bar{F}_{F_a}(t) = -\, F_{F_a}(t)\, \bar{i}_{s_{a_3}}(t)$.

$\bar{F}_{G_{A_a}}$ is the vector function such that if $t \geq 0$, $\bar{F}_{G_{A_a}}(t)$ is the a gear axle force from the axle-wheel interface at the time t.

Each of $F_{G_{A_{a_b}}}$: b = 1, 3 is a simple graph such that if $t \geq 0$,

$$\bar{F}_{G_{A_a}}(t) = F_{G_{A_{a_b}}}(t)\, \bar{j}_{G_{a_c}}(t)\, \delta^{bc} \text{ at the time t.}$$

$\bar{M}_{G_{A_a}}$ is the vector function such that if $t \geq 0$, $\bar{M}_{G_{A_a}}(t)$ is the a gear axle moment from the axle-wheel interface at the time t.

$\hat{M}_{G_{A_{a_1}}}$ and each of $M_{G_{A_{a_b}}}$; b = 1, 3 is a simple graph such that if $t \geq 0$,

$$\bar{M}_{G_{A_a}}(t) = \hat{M}_{G_{A_{a_1}}}(t)\, \bar{I}_{W_{a_1}}(t)$$

$$= M_{G_{A_{a_b}}}(t)\, \bar{j}_{G_{a_c}}(t)\, \delta^{bc}.$$

The additional simple graphs required to determine the shock strut force are defined below:

s_a is the simple graph such that if $t \geq 0$, $s_a(t)$ is the a gear stroke at the time t.

A_{P_a} is the simple graph such that if x is in $[0, s_{C_a}]$, $A_{P_a}(x)$ is the area of the metering pin at the a gear stroke x. $A_{P_a}(x)$ is the cross sectional area in the plane where the orifice plate hole area A_{O_a} is

measured.

A_{N_a} is the simple graph such that if x is in [0, s_{C_a}], $A_{N_a}(x)$ is the a gear net orifice area of the main orifice at the a gear stroke x.

$A_{N_{S_a}}$ is the simple graph such that if x is a number, $A_{N_{S_a}}(x)$ is the a gear net orifice area for snubbing at the snubber orifice differential pressure x.

β_a is the simple graph such that if $t \geq 0$, $\beta_a(t)$ is the volumetric expansion of the hydraulic fluid in the a gear fluid chamber at the time t.

β_{S_a} is the simple graph such that if $t \geq 0$, $\beta_{S_a}(t)$ is the volumetric expansion of the hydraulic fluid in the a gear snubbing chamber at the time t.

V_{H_a} is the simple graph such that if $t \geq 0$, $V_{H_a}(t)$ is the a gear fluid chamber volume at the time t.

$V_{H_{S_a}}$ is the simple graph such that if $t \geq 0$, $V_{H_{S_a}}(t)$ is the a gear snubbing chamber volume at the time t.

p_{A_a} is the simple graph such that if x is in [0, s_{C_a}], $p_{A_a}(x)$ is the pressure in the a gear gas chamber at the a gear stroke x.

p_{T_a} is the simple graph such that if $t \geq 0$, $p_{T_a}(t)$ is the pressure in the a gear fluid chamber at the time t.

p_{H_a} is $p_{T_a} - p_{A_a}[s_a]$

$p_{T_{S_a}}$ is the simple graph such that if $t \geq 0$, $p_{T_{S_a}}(t)$ is the pressure in the a gear snubbing chamber at the time t.

$p_{H_{S_a}}$ is $p_{T_{S_a}} - p_{A_a}[s_a]$ for the type I strut configuration shown in Figure B-1.

$p_{H_{S_a}}$ is $p_{T_{S_a}} - p_{T_a}$ for the type II strut configuration shown in Figure B-3.

$BFU_a{}^b$ is the simple graph such that if s_a is in $[0, s_{C_a}]$, $BFU_a{}^b(s_a)$ is the shock strut upper bearing a gear reference system force coefficient for a unit bth force component on the point A_a of the a gear.

$BFL_a{}^b$ is the simple graph such that if s_a is in $[0, s_{C_a}]$, $BFL_a{}^b(s_a)$ is the a gear reference system lower bearing force coefficient for a unit bth force component on the point A_a of the a gear.

$BMU_a{}^b$ is the simple graph such that if s_a is in $[0, s_{C_a}]$, $BMU_a{}^b(s_a)$ is the a gear reference system upper bearing force coefficient for a unit bth moment component on the axle of the a gear.

$BML_a{}^b$ is the simple graph such that if s_a is in $[0, s_{C_a}]$, $BML_a{}^b(s_a)$ is the a gear reference system lower bearing force coefficient for a unit bth moment component in the a gear reference system on the axle of the a gear.

If the a gear stroke, s_a, is in $[0, s_{C_a}]$ and if $t \geq 0$, the shock strut force , $\bar{F}_{S_a}(t)$, may be expressed in terms of the forces and pressures acting on and in the shock strut. This may be accomplished by isolating the piston of the type I shock strut as seen in Figure B-2. From this figure the following expression for the number $F_{S_a}(t)$ may be derived from a summation of the forces on the piston:

$$F_{S_a}(t) = p_{AMB} A_{A_a} - p_{T_a}(t) (A_{H_a} - A_{P_a}(s_a(t)))$$

$$+ p_{T_{S_a}}(t) (A_{H_{S_a}} - A_{N_{S_a}}(p_{H_{S_a}}(t)))$$

$$- p_{A_a}(s_a(t)) (A_{A_a} + A_{H_{S_a}} - A_{N_{S_a}}(p_{H_{S_a}}(t)) - A_{H_a})$$

$$- p_{A_a}(s_a(t)) A_{P_a}(s_a(t)) + F_{F_a}(t).$$

With

$$p_{T_a}(t) = p_{H_a}(t) + p_{A_a}(s_a(t))$$

and

$$p_{T_{S_a}}(t) = p_{H_{S_a}}(t) + p_{A_a}(s_a(t))$$

it follows that

$$F_{S_a}(t) = p_{AMB} A_{A_a} - p_{H_a}(t) (A_{H_a} - A_{P_a}(s_a(t)))$$

$$+ p_{H_{S_a}}(t) (A_{H_{S_a}} - A_{N_{S_a}}(p_{H_{S_a}}(t))) - p_{A_a}(s_a(t)) A_{A_a} + F_{F_a}(t)$$

where

$$A_{P_a}(t) = A_{O_a} - A_{N_a}(s_a(t)).$$

When the shock strut has bottomed (i.e. the shock strut stroke is less than zero) it is assumed that the bottoming load is derived from a linear spring and a linear (viscous) damper. Thus, for this case

$$F_{S_a}(t) = p_{AMB}\, A_{A_a} - p_{H_a}(t)\,(A_{H_a} - A_{P_a}(s_a(t)))$$

$$+ p_{H_{S_a}}(t)\,(A_{H_{S_a}} - A_{N_{S_a}}(p_{H_{S_a}}(t))) - p_{A_a}(s_a(t))\, A_{A_a} + F_{F_a}(t)$$

$$- K_{BG_a}\, s_a(t) - C_{BG_a}\, s_a'(t).$$

It is now required to derive the equation for the pressure in the shock strut fluid chamber It is supposed that at the time t this pressure is directly related to the volumetric expansion of the hydraulic fluid and inversely proportional to the volume in the fluid chamber.

This relationship may be expressed by

$$p_{T_a}(t) = \frac{-K_H}{V_{H_a}(t)}\,\beta_a(t) \qquad \text{if } \beta_a(t) \text{ is } < 0$$

and

$$p_{T_a}(t) = 0 \qquad \text{if } \beta_a(t) \text{ is } \geq 0.$$

The volumetric expansion at the time t, $\beta_a(t)$, may be obtained from an integration of the sum of the functions that define the fluid volume leaving the fluid chamber through the orifice and the rate of of change of the volume in the fluid chamber. Thus,

$$\beta_a' = C_{D_a} A_{N_a}[s_a] v_{H_a} + V_{H_a}'$$

is the function to be integrated where at the time t the velocity of fluid through the main orifice, $\bar{v}_{H_a}(t)$, is derived from the principle of conservation of energy and the number $v_{H_a}(t)$ is therefore

$$v_{H_a}(t) = \left(2 \frac{p_{H_a}(t)}{\rho_H}\right)^{0.5} \qquad \text{for } p_{H_a}(t) > 0,$$

$$v_{H_a}(t) = 0 \qquad \text{for } p_{H_a}(t) = 0,$$

$$v_{H_a}(t) = -\left(-2 \frac{p_{H_a}(t)}{\rho_H}\right)^{0.5} \qquad \text{for } p_{H_a}(t) < 0,$$

and the rate of change of the volume in the fluid chamber is

$$V_{H_a}(t)' = -(A_{H_a} - A_{P_a}(s_a(t)))\, s_a(t)'.$$

A similar approach is used to determine the pressure at the time t in the snubbing chamber. The required equations are

$$p_{T_{S_a}}(t) = \frac{-K_H}{V_{H_{S_a}}(t)} \beta_{S_a}(t) \qquad \text{if } \beta_{S_a}(t) \text{ is} < 0,$$

$$p_{T_{S_a}}(t) = 0 \qquad \text{if } \beta_{S_a}(t) \geq 0,$$

where

$$\beta_{S_a}(t)' = C_{D_{S_a}} A_{N_{S_a}}(p_{H_{S_a}}(t))\, v_{H_{S_a}}(t) + V_{H_{S_a}}'(t),$$

and

$$v_{H_{S_a}}(t) = \left(2 \frac{p_{H_{S_a}}(t)}{\rho_H}\right)^{0.5} \qquad \text{for } p_{H_{S_a}}(t) > 0,$$

$$v_{H_{S_a}}(t) = 0 \qquad \text{for } p_{H_{S_a}}(t) = 0,$$

$$v_{H_{S_a}}(t) = -\left(-2 \frac{p_{H_{S_a}}(t)}{\rho_H}\right)^{0.5} \qquad \text{for } p_{H_{S_a}}(t) < 0,$$

$$V_{H_{S_a}}(t)' = A_{H_{S_a}} s_a'(t).$$

It is assumed that the gas volume in the shock strut at the time t is defined by

$$V_{A_a}(t) = V_{A_{0_a}} - A_{A_a} s_a(t).$$

It is further assumed that n is a positive number such that the thermodynamic process of the gas in the shock strut gas chamber during compression is represented by

$$p_{A_a}(t) = p_{A_{0_a}} \left(\frac{V_{A_{0_a}}}{V_{A_a}(t)}\right)^n.$$

For the type II shock strut at the time t shown in Figure B-3 the pressures $p_{T_a}(t)$, $p_{T_{S_a}}(t)$ and $p_{A_a}(t)$ are calculated from the relations derived for the type I shock strut. There is, however, a different equilibrium equation from which the number $F_{S_a}(t)$ is derived. The following

expression for $F_{S_a}(t)$ may be derived from Figure B-4:

$$F_{S_a}(t) = p_{AMB}\, A_{A_a} - p_{T_a}(t)\,(A_{H_a} - A_{P_a}(s_a(t)) - A_{N_{S_a}}(p_{H_{S_a}}(t)))$$

$$+ p_{T_{S_a}}(t)\,(A_{H_{S_a}} - A_{N_{S_a}}(p_{H_{S_a}}(t))) - p_{A_a}(s_a(t))\, A_{P_a}(s_a(t)) + F_{F_a}(t) \text{ for } s_a(t) \geq 0,$$

and since

$$p_{T_a}(t) = p_{H_a}(t) + p_{A_a}(s_a(t)),$$

$$p_{T_{S_a}}(t) = p_{H_{S_a}}(t) + p_{T_a}(t),$$

and

$$A_{A_a} = A_{H_a} - A_{H_{S_a}},$$

it follows that

$$F_{S_a} = p_{AMB}\, A_{A_a} - p_{H_a}(t)\,(A_{H_a} - A_{P_a}(s_a(t)) - A_{H_{S_a}})$$

$$+ p_{H_{S_a}}(t)\,(A_{H_{S_a}} - A_{N_{S_a}}(p_{H_{S_a}}(t))) - p_{A_a}(s_a(t))\, A_{A_a} + F_{F_a}(t) \qquad \text{for } s_a(t) \geq 0.$$

For the case where s_a is less than zero, the same stiffness and damping terms that were used for the type I shock strut are to be included.

Also, the rate of change at the time t of the volume in the fluid chamber is

$$V_{H_a}(t)' = - C_{D_{S_a}}\, A_{N_{S_a}}(p_{H_{S_a}}(t))\, v_{H_{S_a}}(t) - (A_{H_a} - A_{P_a}(s_a(t)))\, s_a'(t).$$

To compute the shock strut friction force function it is first neccessary to calculate the force and moment functions for the gear axle. This may be done using Newtonian mechanics. From this method it seen from equation (C-1) that the axle force for the a gear may be expressed by

$$\bar{F}_{G_{A_a}} = - \int_{v_{W_a}} ((\bar{R}_{W_a;I_t})_{;I_t} \rho_{W_a} - \bar{P}) \, dv_{W_a} + \bar{F}_{G_a}.$$

The a gear axle velocity function in the parenthesis of this equation may be found in an expanded form in equation (7) in the Derivation of the Equations of Motion section in the main body of the report.

If the indicated differentiations and integrations are performed and the terms involving the rates of change of gear and airframe deformations are omitted, then the following expression for the numbers $F_{G_{A_{a_b}}}(t)$; b = 1, 3 is obtained:

$$F_{G_{A_{a_b}}}(t) = - M_{W_a} \alpha_{G_{a_b}}{}^{1}(t) \, (V_B{}^{1\prime}(t) + \Omega_B{}^2(t) \, V_B{}^3(t) - \Omega_B{}^3(t) \, V_B{}^2(t))$$

$$- M_{W_a} \alpha_{G_{a_b}}{}^{2}(t) \, (V_B{}^{2\prime}(t) + \Omega_B{}^3(t) \, V_B{}^1(t) - \Omega_B{}^1(t) \, V_B{}^3(t))$$

$$- M_{W_a} \alpha_{G_{a_b}}{}^{3}(t) \, (V_B{}^{3\prime}(t) + \Omega_B{}^1(t) \, V_B{}^2(t) - \Omega_B{}^2(t) \, V_B{}^1(t))$$

$$- M_{W_a} \alpha_{G_{a_b}}{}^{g}(t) \, e^{hcd} \, \Omega_B{}^{p\prime}(t) \, \delta_{cp} \, (1_{T_a}{}^{q}$$

$$+ g_{A_a}{}^{f} \alpha_{G_{a_f}}{}^{q}(t) + \xi_{A_a}{}^{f}(s_a(t)) \, \alpha_{G_{a_f}}{}^{q}(t)) \, \delta_{dq} \, \delta_{gh}$$

$$- M_{W_a} \alpha_{G_{a_b}}{}^{g}(t) \, \delta^{pq}_{rf} \, \Omega_B{}^f(t) \, \Omega_B{}^c(t) \, (1_{T_a}{}^{d}$$

$$+ g_{A_a}{}^{u} \alpha_{G_{a_u}}{}^{d}(t) + \xi_{A_a}{}^{v}(s_a(t)) \, \alpha_{G_{a_v}}{}^{d}(t)) \, \delta_{cp} \, \delta_{dq} \, \delta^{rs} \, \delta_{gs}$$

$$- M_{W_a} \, e^{h3d} \, \Omega_{G_a}{}^{\prime}(t) \, (g_{A_a}{}^{q} + \xi_{A_a}{}^{q}(s_a(t)) \, \delta_{dg} \, \delta_{bh}$$

$$- M_{W_a} \delta_{r3}^{3q} (\Omega_{G_a}(t))^2 (g_{A_a}{}^{d} + \xi_{A_a}{}^{d}(s_a(t))) \delta_{dq} \delta^{rs} \delta_{bs}$$

$$- 2 M_{W_a} \alpha_{G_{a_b}}{}^{g}(t) e^{hcd} \Omega_B{}^{p}(t) \delta_{cp} \xi_{A_a}{}^{f},(s_a(t)) s_a'(t) \alpha_{G_{a_f}}{}^{q}(t) \delta_{dq} \delta_{gh}$$

$$- 2 M_{W_a} e^{h3d} \Omega_{G_a}(t) \xi_{A_a}{}^{q},(s_a(t)) s_a'(t) \delta_{dq} \delta_{bh}$$

$$- M_{W_a} \xi_{A_a}{}^{h}{}''(s_a(t)) (s_a'(t))^2 \delta_{bh} - M_{W_a} \xi_{A_a}{}^{h},(s_a(t)) s_a''(t) \delta_{bh}$$

$$- M_{W_a} \Phi_{BG_{A_{a_r}}}{}^{f}(s_a(t), \theta_{G_a}(t)) q_B{}^{r}{}''(t) \hat{\alpha}_{G_{a_f}}{}^{h}(t) \delta_{bh}$$

$$- M_{W_a} \Phi_{G_{A_{a_r}}}{}^{f}(s_a(t), \theta_{G_a}(t)) q_{G_a}{}^{r}{}''(t) \delta_{bf}$$

$$+ F_{G_a}{}^{g}(t) \gamma_{B_d}{}^{c}(t) \alpha_{G_{a_b}}{}^{d}(t) \delta_{cg} + M_{W_a} g \gamma_{B_d}{}^{3}(t) \alpha_{G_{a_b}}{}^{d}(t).$$

The contribution of the axle moment from the acceleration of the wheel mass typically has a negligible influence on the strut friction forces. Therefore, this moment contribution will not be formulated. However, the axle moment from the ground forces may in some cases have a significant effect on the strut friction force. It is assumed that the axle moment from the ground force is parallel to $\bar{I}_{W_{a_1}}(t)$.

Thus, the axle moment from the ground force is

$$\hat{M}_{G_{A_{a_1}}} \bar{I}_{W_{a_1}}(t) = (\bar{I}_{W_{A_1}}(t) \cdot (\bar{w}_{C_a}(t) + \bar{v}_{W_a}(t)) \times \bar{F}_{G_a}(t)) \bar{I}_{W_{a_1}}(t).$$

Therefore, it is found that

$$\hat{M}_{G_{A_{a_1}}}(t) = \gamma_{W_{a_1}}{}^{b}(t)\ e_{bcd}\ (w_{C_a}{}^{c}(t) + v_{W_a}(t))\ F_{G_a}{}^{d}(t).$$

The moment components at the time t in the a gear reference system are determined from

$$M_{G_{A_{a_b}}}(t) = \hat{M}_{G_{A_{a_1}}}(t)\ \bar{I}_{W_{a_1}}(t) \cdot \bar{J}_{G_{a_b}}(t)$$

or

$$M_{G_{A_{a_b}}}(t) = \hat{M}_{G_{A_{a_1}}}(t)\ \alpha_{G_{a_b}}{}^{c}(t)\ \gamma_{B_c}^{d}(t)\ \gamma_{W_{a_d}}{}^{1}(t).$$

The next step in the derivation of the shock strut friction force is to determine the bearing reactions between the piston and the cylinder. When computing these reactions the assumption will be made that the forces on the piston (and axle) from the acceleration of this mass can be omitted. This permits the bearing reactions to be computed from a simple expression involving the axle forces and moments. Therefore, it follows that the number $F_{F_a}(t)$ can be determined from the following equation:

$$F_{F_a}(t) = -\frac{s_a'(t)}{|s_a'(t)|}(\mu_{BU}\ (BFU_a{}^{b}(s_a(t))\ F_{G_{A_{a_b}}}(t) + BMU_a{}^{c}(s_a(t))\ M_{G_{A_{a_c}}}(t))$$

$$+ \mu_{BL}(BFL_a{}^{d}(s_a(t))\ F_{G_{A_{a_d}}}(t) + BML_a{}^{e}(s_a(t))\ M_{G_{A_{a_e}}}(t))).$$

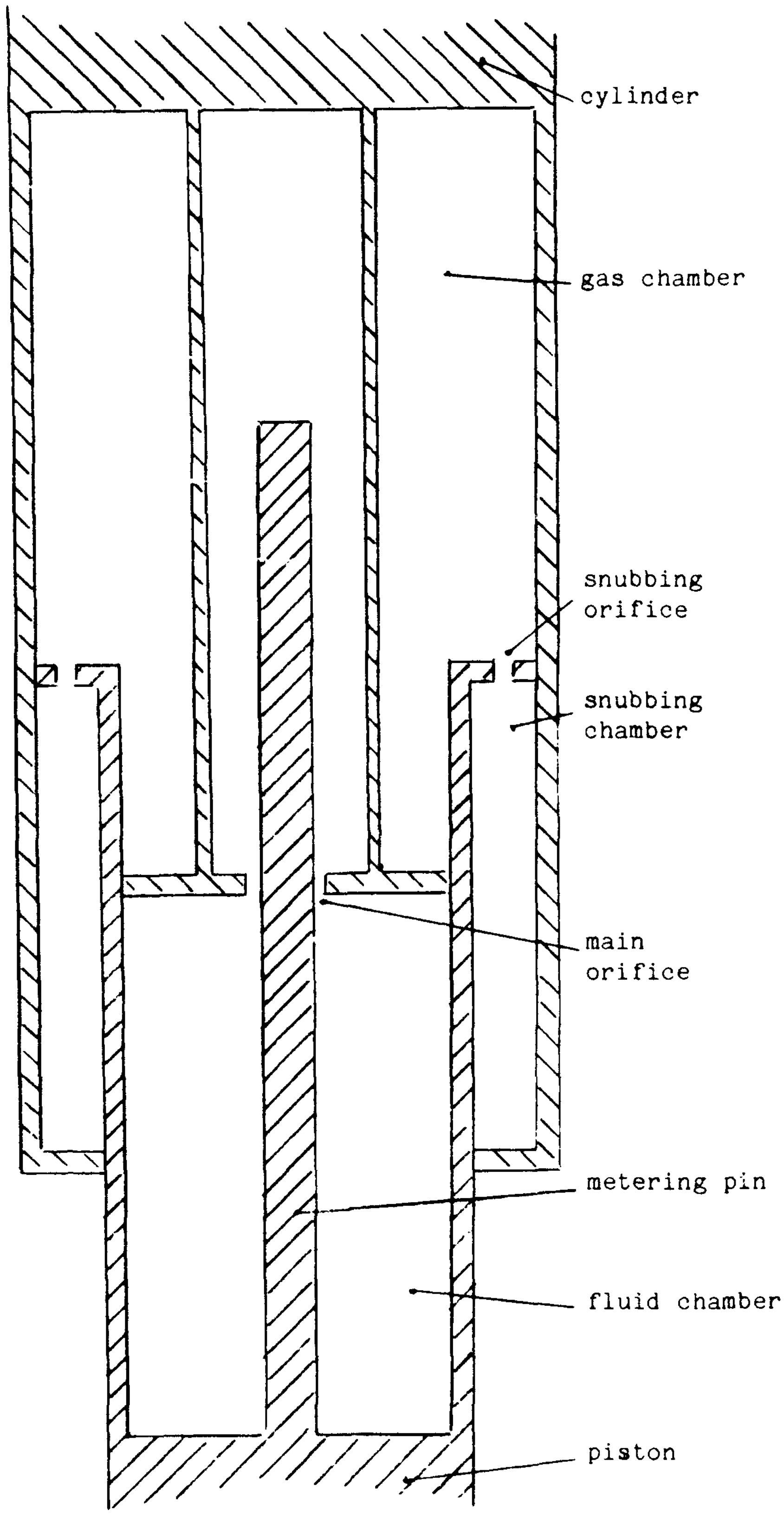


Figure B-1. Schematic of Type I Shock Strut

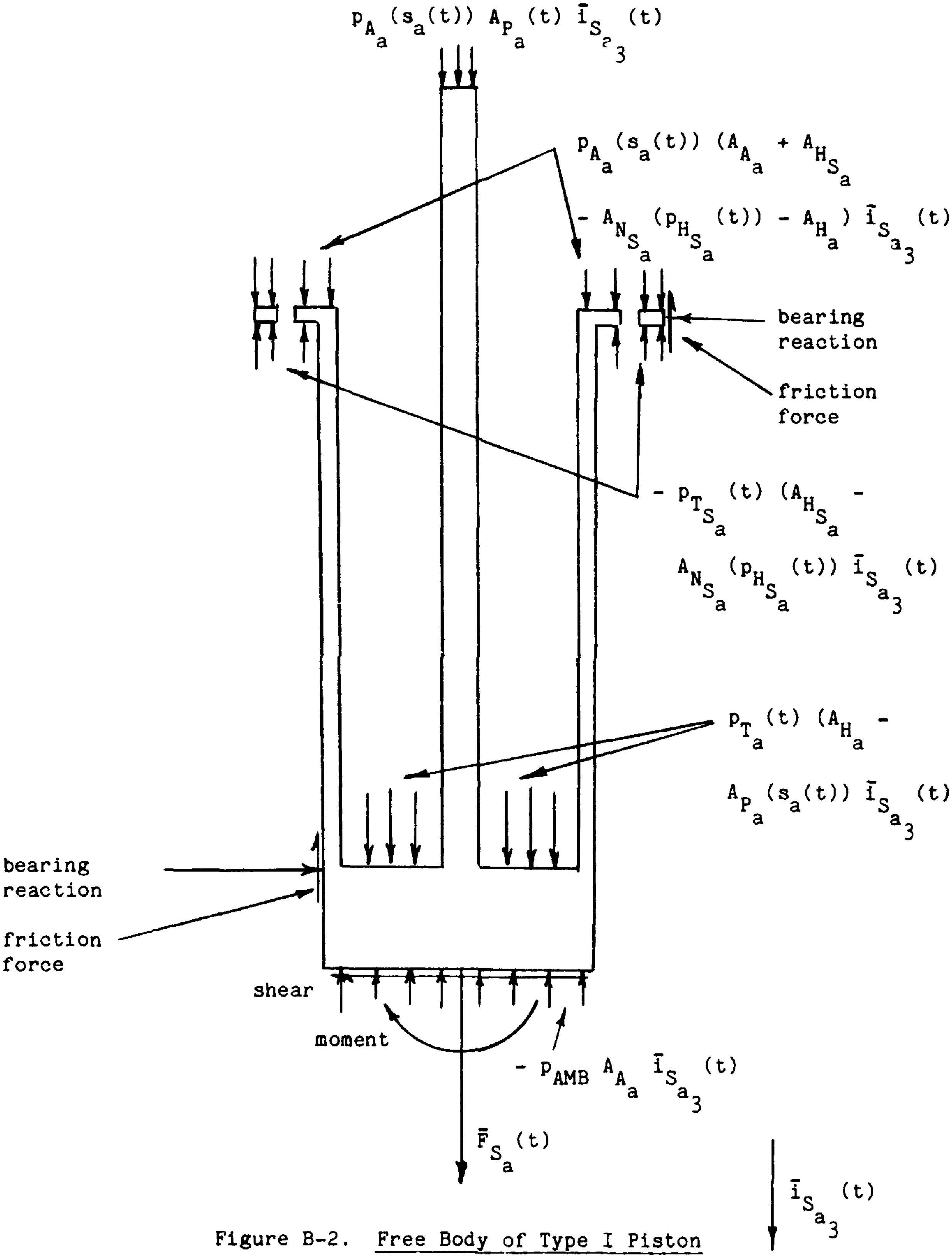


Figure B-2. Free Body of Type I Piston

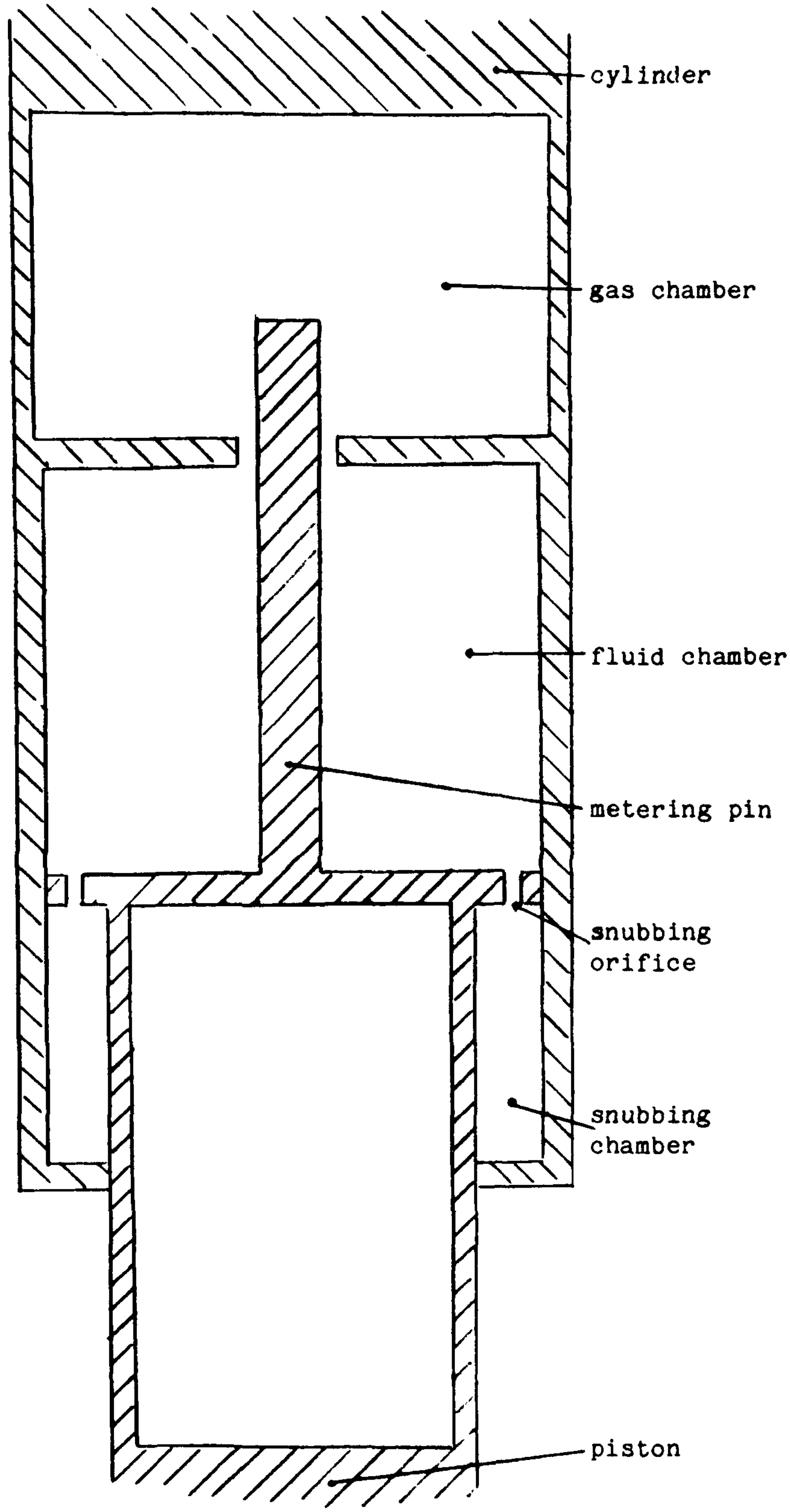


Figure B-3. Schematic of Type II Shock Strut

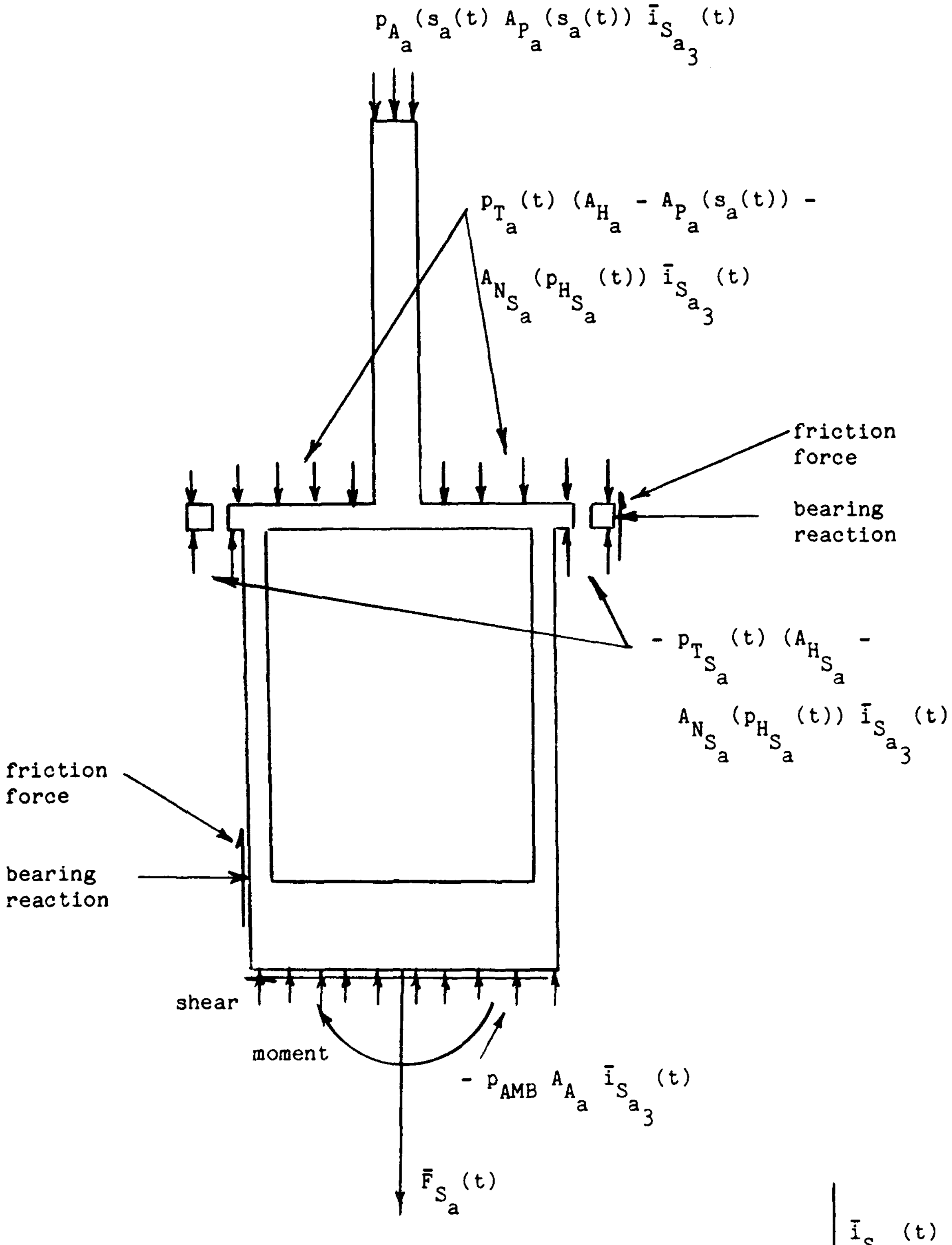


Figure B-4. Free Body of Type II Shock Strut

APPENDIX C

MODIFICATION OF LAGRANGE'S EQUATIONS FOR NON-HOLONOMIC CONSTRAINTS AND QUASI-COORDINATES

Suppose that V is a volume in E^3 space containing a collection of mass particles and that S is the surface of this volume. Further, suppose that Q is a point fixed in space or moving with a constant velocity and that $\bar{r}$ is a vector function such that if $t \geq 0$ and if (x^1, x^2, x^3) is a point in V, $\bar{r}(x^1, x^2, x^3, t)$ is the vector at the time t from Q to the point labeled (x^1, x^2, x^3) in V.

Now suppose that $\delta\bar{r}$ is the vector function such that if $t \geq 0$ and if (x^1, x^2, x^3) is a point in V, $\delta\bar{r}(x^1, x^2, x^3, t)$ is a vector which at the time t is tangent to the constraints (which are assumed to be fixed at the time t) on the point labeled (x^1, x^2, x^3) in V but otherwise arbitrary in magnitude and direction.

Further suppose that

$\bar{P}$ is the vector function such that if $t \geq 0$ and if (x^1, x^2, x^3) is a point in V, $\bar{P}(x^1, x^2, x^3, t)$ is the body force per unit volume at the time t at the point labeled (x^1, x^2, x^3) in V,

$\bar{\nu}$ is the vector function such that if (x^1, x^2, x^3) is a point on the surface of V, $\bar{\nu}(x^1, x^2, x^3)$ is the unit vector normal to the surface at the point labeled (x^1, x^2, x^3) on the surface of V.

$\bar{T}^{\nu}$ is the vector function such that if $t \geq 0$ and if (x^1, x^2, x^3) is a point on the surface of V, $\bar{T}^{\nu}(x^1, x^2, x^3, t)$ is the surface force per unit area at the time t on the surface point (x^1, x^2, x^3) where the normal vector is $\bar{\nu}(x^1, x^2, x^3)$,

ρ is the simple surface such that if (x^1, x^2, x^3) is in V, $\rho(x^1, x^2, x^3)$ is the mass density of the collection of mass particles at the point labeled (x^1, x^2, x^3) in V, and

δW is the simple graph such that if $t \geq 0$, $\delta W(t)$ is the (virtual) work done at the time t by the component of the body and surface forces in the direction of the vector $\delta\bar{r}(t)$ acting through the distance $|\delta\bar{r}(t)|$, then from Reference (1), the Principle of Virtual Work states that

$$\delta W = \int_V (\bar{P} - \rho(\bar{r}_{;I_t})_{;I_t}) \cdot \delta\bar{r}\, dV + \int_S (\bar{T}^{\nu} \cdot \delta\bar{r})\, dS = 0. \qquad \text{(C-1)}$$

The principle of Virtual Work may be used as a basis for the derivation of Lagrange's equations. However, before Lagrange's equations can be derived, the vector function $\delta\bar{r}$ must be rewritten in terms of the coordinates (i.e. functions) that describe the motion of the collection of mass particles.

A vector function that could be used for this purpose is a modification of the velocity vector function. To develop this vector function suppose that n is a positive integer, each of q^a; a = 1, n is generalized coordinate and $\bar{R}$ is a vector function of class C^1 such that if $t \geq 0$ and (x^1, x^2, x^3) is in V,

$$\bar{r}(x^1, x^2, x^3, t) = \bar{R}(x^1, x^2, x^3, t, q^a(t); a = 1, n)$$

at the time t.

The velocity vector is therefore

$$\bar{r}_{;I_t}(x^1, x^2, x^3, t) = \bar{R}_{;q^b}(x^1, x^2, x^3, t, q^a(t); a = 1, n)\ q^{b\prime}(t)$$
$$+ \bar{R}_{;I_t}(x^1, x^2, x^3, t, q^a(t); a = 1, n).$$

In this expression the $\bar{R}_{;I_t}$ term has a contribution to the velocity for the case where the constraints are time dependent. However, for the calculation of $\delta\bar{r}$, this contribution is not included since the constraints are fixed when $\delta\bar{r}$ is determined.

Therefore, the expression

$$\delta\bar{r}(x^1, x^2, x^3, t) = \bar{R}_{;q^b}(x^1, x^2, x^3, t, q^a(t); a = 1, n)\ q^{b\prime}(t) \qquad \text{(C-2)}$$

may be used to derive the vector function $\delta\bar{r}$ that could be used in equation (C-1) where it is understood that (within the constraints) that each of the coordinate derivatives are independent and arbitrary in magnitude.

This substitution for the $\delta\bar{r}$ in equation (C-1) will provide the formulation needed for the derivation of Lagrange's equations for the case where the motion is expressible in terms generalized coordinates.

For this purpose suppose that T is a simple graph such that if $t \geq 0$, T(t) is the kinetic energy of the collection of mass particles at the time

t. Further, suppose that $\hat{K}$ is a simple surface such that

$$T = \hat{K}[q^c;\ c = 1, n,\ q^{c\prime};\ c = 1, n]. \tag{C-3}$$

Therefore, if each of Q_a; a = 1, n is a simple graph such if $t \geq 0$, $Q_a(t)$; a = 1, n are the generalized forces acting on the collection of mass particles at the time t and the acceleration function term is expressed in terms of the kinetic energy function as illustrated in Reference (5), then the Principle of Virtual Work may be rewritten in the form of power rather than work as follows:

$$((\hat{K}_{;q^{a\prime}})' - \hat{K}_{;q^a} - Q_a)\ q^{a\prime} = 0. \tag{C-4}$$

Lagrange's equations may be easily derived from equation (C-4) for the case where the velocity functions are independent.

In some cases it is preferable to write the equations of motion in terms of non-generalized (quasi) coordinate velocity component functions. For example, it is frequently desirable to have the rigid body motion of a collection of mass points be determined from the body axis components of the velocity vector function and the angular velocity vector function. These are not, however, generalized coordinate velocity component functions since the position of the body cannot, in the general case, be determined from an integration of these velocity component functions.

Suppose that each of V^a; a = 1, n is a quasi-coordinate velocity component function and that each of A^a_b; a = 1, n; b = 1, n is a simple surface of class C^1 at each of its points and

$$V^a = A^a_b[q^c;\ c = 1,\ n]\ q^{b\prime}. \tag{C-5}$$

Further, suppose that each of B^a_b; $a = 1, n$; $b = 1, n$ is a simple surface such that

$$q^{a\prime} = B^a_b[q^c;\ c = 1,\ n]\ V^b.$$

It follows then that if

$$\alpha^a_b = A^a_b[q^c;\ c = 1,\ n]$$

and

$$\beta^a_b = B^a_b[q^c;\ c = 1,\ n],$$

then $\alpha^a_b \beta^b_c$ is unity for the case where a is equal to c and zero for the case where a is unequal to c. Also, it is seen that

$$q^{a\prime} = \beta^a_b\ V^b. \tag{C-6}$$

If equation (C-6) is substituted into equation (C-4) and if each of S_b; $b = 1, n$ is a simple graph such that

$$S_b = \beta^a_b\ Q_a, \tag{C-7}$$

then

$$(\beta^a_b\ (\hat{K}_{;q^{a\prime}})' - \beta^a_b\ \hat{K}_{;q^a} - S_b)\ V^b = 0. \tag{C-8}$$

It remains now to replace $\hat{K}_{;q^{a\prime}}$ and $\hat{K}_{;q^a}$ with functions expressible in

terms of quasi-coordinate velocity component functions. For the purpose, suppose that K is a simple surface such that

$$T = K[q^c;\ c = 1,\ n,\ V^c;\ c = 1,\ n]. \tag{C-9}$$

The kinetic energy functions in equations (C-3) and (C-9) may be differentiated to yield

$$\begin{aligned} T' &= \hat{K}_{;q^a}\, q^{a\prime} + \hat{K}_{;q^{a\prime}}\, q^{a''} \\ &= K_{;q^a}\, q^{a\prime} + K_{;V^a}\, V^{a\prime}. \end{aligned} \tag{C-10}$$

But, from equation (C-5)

$$V^{a\prime} = A^a_{b;q^c}\, q^{b\prime}\, q^{c\prime} + A^a_b[q^d;\ d = 1,\ n]\, q^{b''}.$$

Therefore, by comparison of like terms in equation (C-10) it is seen that

$$\hat{K}_{;q^{a\prime}} = K_{;V^d}\, A^d_a[q^c;\ c = 1,\ n],$$

$$\hat{K}_{;q^a} = K_{;q^a} + K_{;V^d}\, A^d_{b;q^a}\, q^{b\prime}.$$

If these expressions are substituted into equation (C-8) then it is found that

$$\beta^a_b[\ {}^d_a\,(K_{;V^d})' + K_{;V^d}\, A^d_{a;q^c}\, \beta^c_e\, V^e$$

$$- K_{;q^a} - K_{;V^d}\, A^d_{c;q^a}\, \beta^c_e\, V^e] = S_b,$$

and if

$$\Omega_b^d = \beta_b^a (A^d_{c;q^a} - A^d_{a;q^c}) \beta_e^c V^e, \tag{C-11}$$

then

$$((K_{;V^b})' - \Omega_b^d K_{;V^d} - \beta_b^a K_{;q^a} - S_b) V^b = 0 \tag{C-12}$$

is the Principle of Virtual Work expressed in terms of quasi-coordinate velocity component functions.

Now suppose that m is a positive integer, but less than n and that each of Γ_{ab}; a = 1, m is a simple surface such that if $t \geq 0$,

$$\Gamma_{ab}(t, q^c(t); c = 1, n) V^b(t) = 0 \tag{C-13}$$

is a constraint on the collection of particles at the time t. Note that the constraint relation is linear in terms of the quasi-coordinate velocity component functions.

Further suppose that $\hat{F}_a$ is a simple surface such that if $t \geq 0$,

$$\begin{aligned} &\hat{F}_a(t, q^c(t); c = 1, n, V^b(t); b = 1, n) \\ &= \Gamma_{ab}(t, q^c(t); c = 1, n) V^b(t) = 0 \end{aligned} \tag{C-14}$$

at the time t and if each of λ^a; a = 1, n is a simple graph (a Lagrangian multiplier) such that

$$\lambda^a(t) \hat{F}_a(t, q^c(t); c = 1, n, V^b(t); b = 1, n) = 0 \tag{C-15}$$

at the time t.

It follows then that for the case of non-holonomic constraints

expressed in the form of equation (C-13), Lagrange's equations may be written as follows:

$$K_{;V^b})' - \Omega^d_b K_{;V^d} - \beta^a_b K_{;q^a} = S_b + \lambda^a \hat{F}_{a;V^b}. \tag{C-16}$$

The use of equation (C-7) to find the S_b; b = 1, n functions is often difficult. A more direct approach is to develop another candidate for $\delta \bar{r}$ in equation (C-1) in terms of the quasi-coordinate velocity component functions. Suppose that $\bar{\Lambda}_b$ is a vector function such that if $t \geq 0$ and (x^1, x^2, x^3) is in V, then the the sum

$$\bar{\Lambda}_b(x^1, x^2, x^3, t, q^a(t); a = 1, n)\ V^b(t) \tag{C-17}$$

is this candidate for $\delta \bar{r}(x^1, x^2, x^3, t)$ at the time t for the point labeled (x^1, x^2, x^3) in V. Note that the vector in (C-17) is the same as the vector in (C-2). However, the vector sum in (C-17) arises naturally from the differentiation of the vector function $\bar{r}$ when quasi-coordinate velocity component functions are used. It follows then from equation (C-1) that

$$S_b = \int_V (\bar{P} \cdot \bar{\Lambda}_b)\, dV + \int_S (\bar{T}^{\nu} \cdot \bar{\Lambda}_b)\, dS. \tag{C-18}$$

APPENDIX D

DEFINITION OF THE MASS TERMS

The mass terms which appear in equation (8) are defined below. These terms are derived from the operations required when equations (5), (6) and (7) are substituted into equation (1).

$$(M^{V_B V_B}{}_{bc})$$

$$= \left(\int_{V_B} \rho_B \, dV_B + \sum_{a=1}^{N_G} \int_{V_{G_a}} \rho_{G_a} \, dV_{G_a} + \sum_{a=1}^{N_G} \int_{V_{W_a}} \rho_{W_a} \, dV_{W_a}\right) \delta_{bc}$$

$$(M^{\Omega_B \Omega_B}{}_{bc})[s_a, \theta_{G_a}; a = 1, N_G]$$

$$= \int_{V_B} \delta^{gh}_{bd} \, L_B{}^d \, L_B{}^e \, \delta_{eh} \, \delta_{cg} \, \rho_B \, dV_B$$

$$+ \sum_{a=1}^{N_G} \int_{V_{G_a}} \delta^{gh}_{bd} \left(1_{T_a}{}^d + \alpha_{G_{a_f}}{}^d \left(G_a{}^f + \Xi_a{}^f[I_{x_{G_a}}1, I_{x_{G_a}}2, I_{x_{G_a}}3, s_a]\right)\right)$$

$$\times \left(1_{T_a}{}^e + \alpha_{G_{a_f}}{}^e \left(G_a{}^f + \Xi_a{}^f[I_{x_{G_a}}1, I_{x_{G_a}}2, I_{x_{G_a}}3, s_a]\right)\right) \delta_{eh} \, \delta_{cg} \, \rho_{G_a} \, dV_{G_a}$$

$$+ \sum_{a=1}^{N_G} \int_{V_{W_a}} \delta^{gh}_{bd} \left(1_{T_a}{}^d + \alpha_{G_{a_f}}{}^d \left(g_{A_a}{}^f + \xi_{A_a}{}^f[s_a] + \beta_{G_{a_m}}{}^f \, W_a{}^m\right)\right)$$

$$\times \left(1_{T_a}{}^e + \alpha_{G_{a_f}}{}^e \left(g_{A_a}{}^f + \xi_{A_a}{}^f[s_a] + \beta_{G_{a_m}}{}^f \, W_a{}^m\right)\right) \delta_{eh} \, \delta_{cg} \, \rho_{W_a} \, dV_{W_a}$$

$$(M^{\Omega_{G_a}\Omega_{G_a}})[s_a]$$

$$= \int_{V_{G_a}} \delta^{3d}_{3c} \, (G_a{}^c + \Xi_a{}^c[I_{x_{G_a}}1, I_{x_{G_a}}2, I_{x_{G_a}}3, s_a])$$

$$\times \, (G_a{}^h + \Xi_a{}^h[I_{x_{G_a}}1, I_{x_{G_a}}2, I_{x_{G_a}}3, s_a]) \, \delta_{dh} \, \rho_{G_a} \, dV_{G_a}$$

$$+ \int_{V_{W_a}} \delta^{3d}_{3c} \, (g_{A_a}{}^c + \xi_{A_a}{}^c[s_a] + \beta_{G_{a_f}}{}^c \, W_a{}^f)$$

$$\times \, (g_{A_a}{}^h + \xi_{A_a}{}^h[s_a] + \beta_{G_{a_f}}{}^h \, W_a{}^f) \, \delta_{dh} \, \rho_{W_a} \, dV_{W_a}$$

$$(M^{\Omega_{W_a}\Omega_{W_a}})$$

$$= \int_{V_{W_a}} \delta^{2r}_{2g} \, W_a{}^f \, W_a{}^g \, \delta_{rf} \, \rho_{W_a} \, dV_{W_a}$$

$$(M^{s_a s_a})[s_a]$$

$$= \int_{V_{G_a}} (\Xi^c_{a;s_a}) \, (\Xi^d_{a;s_a}) \, \delta_{cd} \, \rho_{G_a} \, dV_{G_a}$$

$$+ \int_{V_{W_a}} (\xi_{A_a}{}^{c\prime}[s_a] + B_{G_{a_b}}{}^{c\prime}[s_a] \, W_a{}^b)$$

$$\times \, (\xi_{A_a}{}^{d\prime}[s_a] + B_{G_{a_e}}{}^{d\prime}[s_a] \, W_a{}^e) \, \delta_{cd} \, \rho_{W_a} \, dV_{W_a}$$

$$(M^{q_B}{}_{b}{}^{q_B}{}_{c})[s_a, \theta_{G_a}; a = 1, N_G]$$

$$= \int_{V_B} \Phi_{B_b}{}^{d} \Phi_{B_c}{}^{e} \delta_{de} \rho_B \, dV_B$$

$$+ \sum_{a=1}^{N_G} \int_{V_{G_a}} (\Phi_{BG_{a_b}}{}^{d}[I_{x_{G_a}}{}^{1}, I_{x_{G_a}}{}^{2}, I_{x_{G_a}}{}^{3}, s_a, \theta_{G_a}])$$

$$\times (\Phi_{BG_{a_c}}{}^{e}[I_{x_{G_a}}{}^{1}, I_{x_{G_a}}{}^{2}, I_{x_{G_a}}{}^{3}, s_a, \theta_{G_a}]) \delta_{de} \rho_{G_a} \, dV_{G_a}$$

$$+ \sum_{a=1}^{N_G} \int_{V_{W_a}} \Phi_{BG_{A_{a_b}}}{}^{d}[s_a, \theta_{G_a}] \Phi_{BG_{A_{a_c}}}{}^{e}[s_a, \theta_{G_a}] \delta_{de} \rho_{W_a} \, dV_{W_a}$$

$$(M^{q_{G_a}}{}_{b}{}^{q_{G_a}}{}_{c})[s_a, \theta_{G_a}]$$

$$= \int_{V_{G_a}} (\Phi_{G_{a_b}}{}^{d}[I_{x_{G_a}}{}^{1}, I_{x_{G_a}}{}^{2}, I_{x_{G_a}}{}^{3}, s_a, \theta_{G_a}])$$

$$\times (\Phi_{G_{a_c}}{}^{e}[I_{x_{G_a}}{}^{1}, I_{x_{G_a}}{}^{2}, I_{x_{G_a}}{}^{3}, s_a, \theta_{G_a}]) \delta_{de} \rho_{G_a} \, dV_{G_a}$$

$$+ \int_{V_{W_a}} \Phi_{G_{A_{a_b}}}{}^{d}[s_a, \theta_{G_a}] \Phi_{G_{A_{a_c}}}{}^{e}[s_a, \theta_{G_a}] \delta_{de} \rho_{W_a} \, dV_{W_a}$$

$$({}^{V_B}M_b{}^{\Omega_B}{}_c)[s_a, \theta_{G_a}; a = 1, N_G]$$

$$= \int_{V_B} e_{bcd}\, L_B{}^d\, \rho_B\, dV_B$$

$$+ \sum_{a=1}^{N_G} \int_{V_{G_a}} e_{bcd}\, (1_{T_a}{}^d + \alpha_{G_{a_m}}{}^d\, (G_a{}^m$$

$$+ \Xi_a{}^m[I_{x_{G_a}}1, I_{x_{G_a}}2, I_{x_{G_a}}3, s_a]))\, \rho_{G_a}\, dV_{G_a}$$

$$+ \sum_{a=1}^{N_G} \int_{V_{W_a}} e_{bcd}\, (1_{T_a}{}^d + \alpha_{G_{a_m}}{}^d\, (g_{A_a}{}^m + \xi_{A_a}{}^m[s_a] + \beta_{G_{a_f}}{}^m\, W_a{}^f)\, \rho_{W_a}\, dV_{W_a}$$

$$({}^{V_B}M_b{}^{\Omega_{G_a}})[s_a, \theta_{G_a}]$$

$$= \int_{V_{G_a}} e_{h3c}\, \alpha_{G_{a_d}}{}^f\, (G_a{}^c + \Xi_a{}^c[I_{x_{G_a}}1, I_{x_{G_a}}2, I_{x_{G_a}}3, s_a])\, \delta^{dh}\, \delta_{bf}\, \rho_{G_a}\, dV_{G_a}$$

$$+ \int_{V_{W_a}} e_{h3c}\, (\alpha_{G_{a_d}}{}^f\, (g_{A_a}{}^c + \xi_{A_a}{}^c[s_a] + \beta_{G_{a_m}}{}^c\, W_a{}^m))\, \delta^{dh}\, \delta_{bf}\, \rho_{W_a}\, dV_{W_a}$$

$$({}^{V_B}M_b{}^{s_a})[s_a, \theta_{G_a}]$$

$$= \int_{V_{G_a}} \alpha_{G_{a_c}}{}^d\, \Xi^c_{a;s_a}\, \delta_{bd}\, \rho_{G_a}\, dV_{G_a}$$

$$+ \int_{V_{W_a}} \alpha_{G_{a_c}}{}^d\, (\xi_{A_a}{}^{c\prime}[s_a] + B_{G_{a_e}}{}^{c\prime}[s_a]\, W_a{}^e)\, \delta_{bd}\, \rho_{W_a}\, dV_{W_a}$$

$$(M^{V_B}{}_b{}^{q_B}{}_c)[s_a,\ \theta_{G_a};\ a = 1,\ N_G]$$

$$= \int_{V_B} \Phi_{B_c}{}^d\ \delta_{bd}\ \rho_B\ dV_B$$

$$+ \sum_{a=1}^{N_G} \int_{V_{G_a}} \gamma_{G_{a_d}}{}^f\ \Phi_{BG_{a_c}}{}^d[I_{x_{G_a}}1,\ I_{x_{G_a}}2,\ I_{x_{G_a}}3,\ s_a,\ \theta_{G_a}]\ \delta_{bf}\ \rho_{G_a}\ dV_{G_a}$$

$$+ \sum_{a=1}^{N_G} \int_{V_{W_a}} \gamma_{G_{a_d}}{}^f\ \Phi_{BG_{A_{a_c}}}{}^d[s_a,\ \theta_{G_a}]\ \delta_{bf}\ \rho_{W_a}\ dV_{W_a}$$

$$(M^{V_B}{}_b{}^{q_{G_a}}{}_c)[s_a,\ \theta_{G_a}]$$

$$= \int_{V_{G_a}} \alpha_{G_{a_e}}{}^d\ \Phi_{G_{a_c}}{}^e[I_{x_{G_a}}1,\ I_{x_{G_a}}2,\ I_{x_{G_a}}3,\ s_a,\ \theta_{G_a}]\ \delta_{bd}\ \rho_{G_a}\ dV_{G_a}$$

$$+ \int_{V_{W_a}} \alpha_{G_{a_e}}{}^d\ \Phi_{G_{a_c}}{}^e[s_a,\ \theta_{G_a}]\ \delta_{bd}\ \rho_{W_a}\ dV_{W_a}$$

$$(M^{\Omega_B}{}_b{}^{\Omega_{G_a}})[s_a,\ \theta_{G_a}]$$

$$= \int_{V_{G_a}} \delta_{bc}^{nq}\ (1_{T_a}{}^c + \alpha_{G_{a_h}}{}^c\ (G_a{}^h + \Xi_a{}^h[I_{x_{G_a}}1,\ I_{x_{G_a}}2,\ I_{x_{G_a}}3,\ s_a]))$$

$$\times\ \alpha_{G_{a_3}}{}^r\ (\alpha_{G_{a_d}}{}^s\ (G_a{}^d + \Xi_a{}^d[I_{x_{G_a}}1,\ I_{x_{G_a}}2,\ I_{x_{G_a}}3,\ s_a]))\ \delta_{nr}\ \delta_{qs}\ \rho_{G_a}\ dV_{G_a}$$

$$+ \int_{V_{W_a}} \delta_{bc}^{nq}\ (1_{T_a}{}^c + \alpha_{G_{a_h}}{}^c\ (g_{A_a}{}^h + \xi_{A_a}{}^h[s_a] + \beta_{G_{a_m}}{}^h\ W_a{}^m))$$

$$\times\ \alpha_{G_{a_3}}{}^r\ \alpha_{G_{a_d}}{}^s\ (g_{A_a}{}^d + \xi_{A_a}{}^d[s_a] + \beta_{G_{a_f}}{}^d\ W_a{}^f)\ \delta_{nr}\ \delta_{qs}\ \rho_{W_a} dV_{W_a}$$

$$\left(M_{b}^{\Omega_B \Omega_{W_a}}\right)[s_a, \theta_{G_a}]$$

$$= \int_{V_{W_a}} \delta_{bc}^{nq} \, (1_{T_a}{}^{c} + \alpha_{G_{a_h}}{}^{c} \, (g_{A_a}{}^{h}$$

$$+ \xi_{A_a}{}^{h}[s_a] + \beta_{G_{a_m}}{}^{h} \, W_a{}^{m})) \, \beta_{G_{a_2}}{}^{p} \, \alpha_{G_{a_p}}{}^{r} \, \beta_{G_{a_d}}{}^{h} \, \alpha_{G_{a_h}}{}^{s} \, W_a{}^{d} \, \delta_{nr} \, \delta_{qs} \, \rho_{W_a} \, dV_{W_a}$$

$$\left(M_{b}^{\Omega_B s_a}\right)[s_a, \theta_{G_a}]$$

$$= \int_{V_{G_a}} e_{hbc} \, (1_{T_a}{}^{c} + \alpha_{G_{a_m}}{}^{c} \, (G_a{}^{m}$$

$$+ \Xi_a{}^{m}[I_{x_{G_a}}1, I_{x_{G_a}}2, I_{x_{G_a}}3, s_a])) \, \alpha_{G_{a_d}}{}^{g} \, \Xi_{a;s_a}^{d} \, \delta_g^h \, \rho_{G_a} dV_{G_a}$$

$$+ \int_{V_{W_a}} e_{hbc} \, (1_{T_a}{}^{c} + \alpha_{G_{a_m}}{}^{c} \, (g_{A_a}{}^{m} + \xi_{A_a}{}^{m}[s_a] + \beta_{G_{a_f}}{}^{m} \, W_a{}^{f}))$$

$$\times \, \alpha_{G_{a_d}}{}^{g} \, (\xi_{A_a}{}^{d,}[s_a] + B_{G_{a_e}}{}^{d,}[s_a] \, W_a{}^{e}) \, \delta_g^h \, \rho_{W_a} \, dV_{W_a}$$

$$(M^{\Omega_B}{}_{b}{}^{q_B}{}_{c})[s_a, \theta_{G_a}; a = 1, N_G]$$

$$= \int_{V_B} e_{hbs} \, L_B{}^s \Phi_{B_c}{}^f \delta^h_f \, \rho_B \, dV_B$$

$$+ \sum_{a=1}^{N_G} \int_{V_{G_a}} e_{hbs} \, (1_{T_a}{}^s + \alpha_{G_{a_m}}{}^s \, (G_a{}^m + \Xi_a{}^m[I_{x_{G_a}}1, I_{x_{G_a}}2, I_{x_{G_a}}3, s_a]))$$

$$\times \, \gamma_{G_{a_n}}{}^f \Phi_{BG_{a_c}}{}^n[I_{x_{G_a}}1, I_{x_{G_a}}2, I_{x_{G_a}}3, s_a, \theta_{G_a}] \, \delta^h_f \, \rho_{G_a} \, dV_{G_a}$$

$$+ \sum_{a=1}^{N_G} \int_{V_{W_a}} e_{hbs} \, (1_{T_a}{}^s + \alpha_{G_{a_m}}{}^s(g_{A_a}{}^m + \xi_{A_a}{}^m[s_a] + \beta_{G_{a_r}}{}^m \, W_a{}^r)$$

$$\times \, \gamma_{G_{a_n}}{}^f \Phi_{BG_{A_{a_c}}}{}^n[s_a, \theta_{G_a}] \, \delta^h_f \, \rho_{W_a} \, dV_{W_a}$$

$$(M^{\Omega_B}{}_{b}{}^{q_{G_a}}{}_{c})[s_a, \theta_{G_a}]$$

$$= \int_{V_{G_a}} e_{hbs} \, (1_{T_a}{}^s + \alpha_{G_{a_m}}{}^s \, (G_a{}^m + \Xi_a{}^m[I_{x_{G_a}}1, I_{x_{G_a}}2, I_{x_{G_a}}3, s_a]))$$

$$\times \, \alpha_{G_{a_n}}{}^f \Phi_{G_{a_c}}{}^n[I_{x_{G_a}}1, I_{x_{G_a}}2, I_{x_{G_a}}3, s_a, \theta_{G_a}] \, \delta^h_f \, \rho_{G_a} \, dV_{G_a}$$

$$+ \int_{V_{W_a}} e_{hbs} \, (1_{T_a}{}^s + \alpha_{G_{a_m}}{}^s \, (g_{A_a}{}^m + \xi_{A_a}{}^m[s_a] + \beta_{G_{a_g}}{}^m \, W_a{}^g))$$

$$\times \, \alpha_{G_{a_n}}{}^f \Phi_{G_{A_{a_c}}}{}^n[s_a, \theta_{G_a}] \, \delta^h_f \, \rho_{W_a} \, dV_{W_a}$$

$$(M^{\Omega_{G_a}\Omega_{W_a}})[s_a]$$

$$= \int_{V_{W_a}} \delta^{3g}_{hf} \, (g_{A_a}{}^{b}$$

$$+ \xi_{A_a}{}^{b}[s_a] + \beta_{G_{a_m}}{}^{b} \, W_a{}^{m}) \, \beta_{G_{a_2}}{}^{h} \, \beta_{G_{a_d}}{}^{f} \, W_a{}^{d} \, \delta_{gb} \, \rho_{W_a} \, dV_{W_a}$$

$$(M^{\Omega_{G_a} s_a})[s_a]$$

$$= \int_{V_{G_a}} e_{h3c} \, (G_a{}^{c} + \Xi_a{}^{c}[I_{x_{G_a}}1, \; I_{x_{G_a}}2, \; I_{x_{G_a}}3, \; s_a]) \, \Xi^{d}_{a;s_a} \, \delta^{h}_{d} \, \rho_{G_a} \, dV_{G_a}$$

$$+ \int_{V_{W_a}} e_{h3c} \, (g_{A_a}{}^{c}$$

$$+ \xi_{A_a}{}^{c}[s_a] + \beta_{G_{a_m}}{}^{c} \, W_a{}^{m}) \, (\xi_{A_a}{}^{d\prime}[s_a] + B_{G_{a_e}}{}^{d\prime}[s_a] \, W_a{}^{e}) \, \delta^{h}_{d} \, \rho_{W_a} \, dV_{W_a}$$

$$(M^{\Omega_{G_a} q_{B_b}})[s_a, \; \theta_{G_a}]$$

$$= \int_{V_{G_a}} e_{h3c} \, (G_a{}^{c} + \Xi_a{}^{c}[I_{x_{G_a}}1, \; I_{x_{G_a}}2, \; I_{x_{G_a}}3, \; s_a]$$

$$\times \; \hat{\alpha}_{G_{a_d}}{}^{e} \, \Phi_{BG_{a_b}}{}^{d}[I_{x_{G_a}}1, \; I_{x_{G_a}}2, \; I_{x_{G_a}}3, \; s_a, \; \theta_{G_a}] \, \delta^{h}_{e} \, \rho_{G_a} \, dV_{G_a}$$

$$+ \int_{V_{W_a}} e_{h3c} \, (g_{A_a}{}^{c}$$

$$+ \xi_{A_a}{}^{c}[s_a] + \beta_{G_{a_m}}{}^{c} \, W_a{}^{m}) \, \hat{\alpha}_{G_{a_d}}{}^{e} \, \Phi_{BG_{A_{a_b}}}{}^{d}[s_a, \; \theta_{G_a}] \, \delta^{h}_{e} \, \rho_{W_a} \, dV_{W_a}$$

$$
\begin{aligned}
&(M^{\Omega_{G_a} q_{G_a}}{}_{b})[s_a,\ \theta_{G_a}] \\
&= \int_{V_{G_a}} e_{h3c}\, (G_a{}^{c} + \Xi_a{}^{c}[I_{x_{G_a}}{}^{1},\ I_{x_{G_a}}{}^{2},\ I_{x_{G_a}}{}^{3},\ s_a] \\
&\quad \times \Phi_{G_{a_b}}{}^{d}[I_{x_{G_a}}{}^{1},\ I_{x_{G_a}}{}^{2},\ I_{x_{G_a}}{}^{3},\ s_a,\ \theta_{G_a}])\ \delta^{h}_{d}\ \rho_{G_a}\ dV_{G_a} \\
&\quad + \int_{V_{W_a}} e_{h3c}\, (g_{A_a}{}^{c} + \xi_{A_a}{}^{c}[s_a] + \beta_{G_{a_m}}{}^{c}\, W_a{}^{m})\ \Phi_{G_{A_{a_b}}}{}^{d}[s_a,\ \theta_{G_a}]\ \delta^{h}_{d}\ \rho_{W_a}\ dV_{W_a}
\end{aligned}
$$

$$
\begin{aligned}
&(M^{s_a q_B}{}_{b})[s_a,\ \theta_{G_a}] \\
&= \int_{V_{G_a}} \Xi^{c}_{a;s_a}\ \hat{\alpha}_{G_{a_d}}{}^{e}\ \Phi_{BG_{a_b}}{}^{d}[I_{x_{G_a}}{}^{1},\ I_{x_{G_a}}{}^{2},\ I_{x_{G_a}}{}^{3},\ s_a,\ \theta_{G_a}]\ \delta_{ce}\ \rho_{G_a}\ dV_{G_a} \\
&\quad + \int_{V_{W_a}} (\xi_{A_a}{}^{c\prime}[s_a] + B_{G_{a_f}}{}^{c}[s_a]\ W_a{}^{f})\ \hat{\alpha}_{G_{a_d}}{}^{e}\ \Phi_{BG_{A_{a_b}}}{}^{d}[s_a,\ \theta_{G_a}]\ \delta_{ce}\ \rho_{W_a}\ dV_{W_a}
\end{aligned}
$$

$$
\begin{aligned}
&(M^{s_a q_{G_a}}{}_{b})[s_a,\ \theta_{G_a}] \\
&= \int_{V_{G_a}} \Xi^{c}_{a;s_a}\ \Phi_{G_{a_b}}{}^{d}[I_{x_{G_a}}{}^{1},\ I_{x_{G_a}}{}^{2},\ I_{x_{G_a}}{}^{3},\ s_a,\ \theta_{G_a}]\ \delta_{cd}\ \rho_{G_a}\ dV_{G_a} \\
&\quad + \int_{V_{W_a}} (\xi_{A_a}{}^{c\prime}[s_a] + B_{G_{a_f}}{}^{c\prime}[s_a]\ W_a{}^{f})\ \Phi_{G_{A_{a_b}}}{}^{d}[s_a,\ \theta_{G_a}]\ \delta_{cd}\ \rho_{W_a}\ dV_{W_a}
\end{aligned}
$$

$$(M^{q_B}{}_{b}{}^{q_{G_a}}{}_{c})[s_a,\ \theta_{G_a}]$$

$$= \int_{V_{G_a}} \hat{\alpha}_{G_{a_f}}{}^{d}\, \Phi_{BG_{a_b}}{}^{f}[I_{x_{G_a}}{}^{1},\ I_{x_{G_a}}{}^{2},\ I_{x_{G_a}}{}^{3},\ s_a,\ \theta_{G_a}]$$

$$\times\ \Phi_{G_{a_c}}{}^{e}[I_{x_{G_a}}{}^{1},\ I_{x_{G_a}}{}^{2},\ I_{x_{G_a}}{}^{3},\ s_a,\ \theta_{G_a}]\ \delta_{de}\ \rho_{G_a}\ dV_{G_a}$$

$$+ \int_{V_{W_a}} \hat{\alpha}_{G_{a_f}}{}^{d}\, \Phi_{BG_{A_{a_b}}}{}^{f}[s_a,\ \theta_{G_a}]\, \Phi_{G_{A_{a_c}}}{}^{e}[s_a,\ \theta_{G_a}]\ \delta_{de}\ \rho_{W_a}\ dV_{W_a}$$

APPENDIX E

DERIVATION OF THE GROUND FORCE EQUATIONS

The derivation for the component of the ground force on the tire that is normal to the ground plane was given in the body of the report. It was established as being functional dependent only on the simple graph v_{W_a}. The ground force components in the ground plane are somewhat more complicated. One of the complications is that the ground forces are dependent on whether or not the tire footprint is slipping on the ground surface. Prior to "spin up" of the wheel the tire footprint is slipping on the ground surface and the force in the ground plane is in the direction opposite to the velocity vector of the tire footprint. The magnitude of this force is determined by multiplying the magnitude of the ground force component normal to the ground plane by the friction coefficient which may be defined in the following manner.

Suppose first that R_a is the simple graph such that if $t \geq 0$, $R_a(t)$ is the slip ratio of the a gear tire at the time t (i.e. the ratio of the magnitude of the tire footprint skidding velocity to the magnitude of the velocity of the axle parallel to the ground). Also suppose that μ_{G_a} is the simple graph such that if $x \geq 0$, $\mu_{G_a}(x)$ is the friction coefficient between the a gear tire and the ground at the slip ratio x.

After spin up (i.e. the slip ratio is zero) the forces on the tire in the ground plane change their functional dependence. The drag force component on the tire in the ground plane is assumed to be a number times the normal ground force component. The side (or cornering) force component

is defined through the following functions.

Suppose that ν_{C_a} is the simple graph such that if $t \geq 0$, $\nu_{C_a}(t)$ is the cornering angle of the a gear tire at the time t. Suppose further that F_{C_a} is a simple surface such that the point $(x, y, F_{C_a}(x,y))$ belongs to F_{C_a} only if at a cornering angle x and a normal ground force component y, $F_{C_a}(x, y)$ is the side force component on the tire.

Also, at a time t after spin up of the a gear wheel the component of the ground force parallel to the unit vector $\bar{I}_{W_{a_1}}(t)$ (i.e. the drag force component on the tire) is determined by multiplying the normal ground force component by the rolling resistance μ_{R_a} which is assumed to be a number.

The vector functions needed to define the kinematic relations are described as follows:

$\bar{r}_{A_a}'$ is the vector function such that if $t \geq 0$, $\bar{r}_{A_a}'(t)$ is the velocity of the a gear axle point A_a with respect to the point Q at the time t.

$\bar{r}_{FA_a}'$ is the vector function such that if $t \geq 0$, $\bar{r}_{FA_a}'(t)$ is the velocity of the a gear tire footprint reference point C_{F_a} with respect to the point A_a at the time t.

Each of $\bar{I}_{W_{a_b}}$; b = 1, 3 is a unit vector function such that if $t \geq 0$, $\bar{I}_{W_{a_3}}(t)$ is $\bar{I}_3$, $\bar{I}_{W_{a_1}}(t)$ is the cross product of $\bar{k}_{G_{a_2}}(t)$ (see Appendix G) and $\bar{I}_{W_{a_3}}(t)$, and $\bar{I}_{W_{a_2}}(t)$ is $\bar{I}_{W_{a_3}}(t) \times \bar{I}_{W_{a_1}}(t)$ at the time t.

The derivation of the kinematic relations may now be accomplished with the help of the unit vector systems and the transformations defined in the main body of the report and Appendix G.

The a gear right hand orthogonal unit vector functions $\bar{I}_{W_{a_b}}$; b = 1, 3 for the wheel-ground reference system may be expressed by the following functional relations:

$$\bar{I}_{W_{a_1}} = \frac{\bar{k}_{G_{a_2}} \times \bar{I}_3}{|\bar{k}_{G_{a_2}} \times \bar{I}_3|}$$

$$= \frac{\beta_{G_{a_2}}{}^{b} \alpha_{G_{a_b}}{}^{c} (\gamma_{B_c}{}^{2} \bar{I}_1 - \gamma_{B_c}{}^{1} \bar{I}_2)}{[(\beta_{G_{a_2}}{}^{b} \alpha_{G_{a_b}}{}^{c} \gamma_{B_c}{}^{1})^2 + (\beta_{G_{a_2}}{}^{b} \alpha_{G_{a_b}}{}^{c} \gamma_{B_c}{}^{2})^2]^{0.5}}.$$

$$\bar{I}_{W_{a_2}} = \frac{\bar{I}_3 \times (\bar{k}_{G_{a_2}} \times \bar{I}_3)}{|\bar{I}_3 \times (\bar{k}_{G_{a_2}} \times \bar{I}_3)|}$$

$$= \frac{\beta_{G_{a_2}}{}^{b} \alpha_{G_{a_b}}{}^{c} (\gamma_{B_c}{}^{1} \bar{I}_1 + \gamma_{B_c}{}^{2} \bar{I}_2)}{[(\beta_{G_{a_2}}{}^{b} \alpha_{G_{a_b}}{}^{c} \gamma_{B_c}{}^{1})^2 + (\beta_{G_{a_2}}{}^{b} \alpha_{G_{a_b}}{}^{c} \gamma_{B_c}{}^{2})^2]^{0.5}}.$$

$$\bar{I}_{W_{a_3}} = \bar{I}_3.$$

The axle velocity of the a gear may be obtained from the terms in equation (7) in the main body of the report. This velocity is expressed by

$$\bar{r}_{A_a}'(t) = \bar{V}_B(t) + \bar{\Omega}_B(t) \times (\bar{l}_{T_a}(t) + \bar{g}_{A_a}(t) + \bar{\xi}_{A_a}(s_a(t), t))$$
$$+ \bar{\Omega}_{G_a}(t) \times (\bar{g}_{A_a}(t) + \bar{\xi}_{A_a}(s_a(t), t))$$
$$+ \bar{\Phi}_{BG_{A_{a_b}}}(s_a(t), \theta_{G_a}(t), t)\, q_B{}^{b}{}'(t) + \bar{\xi}_{A_a;s_a}(s_a(t), t)\, s_a'(t)$$
$$+ \bar{\Phi}_{G_{A_{a_b}}}(s_a(t), \theta_{G_a}(t), t)\, q_{G_a}{}^{b}{}'(t),$$

or in expanded form

$$\bar{r}_{A_a}'(t) = \bar{i}_{B_f}(t)\, V_B{}^{f}(t) + \bar{i}_{B_f}(t)\, e^{fph}\, \Omega_B{}^{e}(t)\, l_{T_a}{}^{d}\, \delta_{ep}\, \delta_{dh}$$
$$+ \bar{i}_{B_f}(t)\, e^{fph}\, \Omega_B{}^{e}(t)\, \alpha_{G_{a_c}}{}^{d}(t)\, (g_{A_a}{}^{c} + \xi_{A_a}{}^{c}(s_a(t)))\, \delta_{ep}\, \delta_{dh}$$
$$+ \bar{j}_{G_{a_d}}(t)\, e^{d3f}\, \Omega_{G_a}(t)\, (g_{A_a}{}^{c} + \xi_{A_a}{}^{c}(s_a(t)))\, \delta_{cf}$$
$$+ \bar{i}_{G_{a_d}}(t)\, \Phi_{BG_{A_{a_f}}}{}^{d}(s_a(t), \theta_{G_a}(t))\, q_B{}^{f}{}'(t)$$
$$+ \bar{j}_{G_{a_d}}(t)\, \xi_{A_a}{}^{d}{}'(s_a(t))\, s_a'(t)$$
$$+ \bar{j}_{G_{a_d}}(t)\, \Phi_{G_{A_{a_b}}}{}^{d}(s_a(t), \theta_{G_a}(t))\, q_{G_a}{}^{b}{}'(t).$$

Thus, with

$$\bar{I}_b = \gamma_{W_{a_b}}{}^{d}(t)\, \bar{I}_{W_{a_c}}(t),$$

the axle velocity is

$$\bar{r}_{A_a}'(t) = \bar{I}_{W_{a_c}}(t)\ (\gamma_{B_f}^{\ b}(t)\ \gamma_{W_{a_b}}^{\ c}(t)\ (v_B^{\ f}(t) + e^{fph}\,\Omega_B^{\ e}(t)\ (1_{T_a}^{\ d}$$

$$+ \alpha_{G_{a_q}}^{\ d}(t)\ (g_{A_a}^{\ q} + \xi_{A_a}^{\ q}(s_a(t)))\ \delta_{ep}\ \delta_{dh}$$

$$+ \alpha_{G_{a_d}}^{\ b}(t)\ \gamma_{B_b}^{\ h}(t)\ \gamma_{W_{a_h}}^{\ c}(t)\ e^{d3f}\,\Omega_{G_a}(t)\ (g_{A_a}^{\ q} + \xi_{A_a}^{\ q}(s_a(t)))\ \delta_{qf}$$

$$+ \gamma_{G_{a_d}}^{\ b}\ \gamma_{B_b}^{\ h}(t)\ \gamma_{W_{a_h}}^{\ c}(t)\,\Phi_{BG_{a_f}}^{\ d}(s_a(t),\ \theta_{G_a}(t))\ q_B^{\ f}{}'(t)$$

$$+ \alpha_{G_{a_d}}^{\ b}(t)\ \gamma_{B_b}^{\ h}(t)\ \gamma_{W_{a_h}}^{\ c}(t)\ \xi_{A_a}^{\ d}{}'(s_a(t))\ s_a'(t)$$

$$+ \alpha_{G_{a_d}}^{\ b}(t)\ \gamma_{B_b}^{\ h}(t)\ \gamma_{W_{a_h}}^{\ c}(t)\,\Phi_{G_{A_{a_b}}}^{\ d}(s_a(t),\ \theta_{G_a}(t)))\ q_{G_a}^{\ b}{}'(t)).$$

With the assumption that the stroking velocity has a negligible effect on the velocity $\bar{r}_{FA_a}'(t)$, this vector may be approximated by

$$\bar{r}_{FA_a}'(t) = (\bar{I}_{W_{a_1}}(t)\ (\bar{\Omega}_B(t) \times \bar{w}_{C_a}(t) + (\bar{\Omega}_B(t) \cdot \bar{k}_{G_{a_2}}(t))\ \bar{k}_{G_{a_2}}(t) \times \bar{v}_{W_a}(t)$$

$$+ \bar{\Omega}_{G_a}(t) \times \bar{w}_{C_a}(t) + \bar{\Omega}_{W_a}(t) \times \bar{w}_{C_a}(t) + \bar{\Omega}_{W_a}(t) \times \bar{v}_{W_a}(t)) \cdot \bar{I}_{W_{a_1}}(t))$$

$$+ (\bar{I}_{W_{A_2}}(t)\ (\bar{\Omega}_B(t) \times \bar{w}_{C_a}(t) + \bar{\Omega}_{G_a}(t) \times \bar{w}_{C_a}(t)) \cdot \bar{I}_{W_{a_2}}(t))$$

$$+ \bar{I}_{W_{a_3}}(t)\ v_{W_a}'(t).$$

After the indicated vector operations are performed it is found that

$$\bar{r}_{FA_a}'(t) = \bar{I}_{W_{a_1}}(t)\ ((\Omega_B^{\ b}(t)\ w_{C_a}^{\ c}(t)\ e^{fgh}\ \gamma_{B_b}^{\ d}(t)\ \delta_{dg}\ \delta_{ch}\ \gamma_{W_{a_r}}^{\ 1}(t)\ \delta_f^{\ r})$$

$$+ (\Omega_B^{\ b}(t)\ \beta_{G_{a_2}}^{\ c}(t)\ \alpha_{G_{a_c}}^{\ d}(t)\ \delta_{bd})$$

$$\times\ (v_{W_a}(t)\ \beta_{G_{a_2}}^{\ f}(t)\ \alpha_{G_{a_f}}^{\ g}(t)\ \gamma_{B_g}^{\ h}(t)\ e^{rs3}\ \delta_{hs}\ \gamma_{W_{a_m}}(t)\ \delta_r^{\ m})$$

$$+ (\Omega_{G_a}(t)\ w_{C_a}{}^c(t)\ a_{G_{a_3}}{}^d(t)\ \gamma_{B_d}{}^f(t)\ e^{rst}\ \delta_{fs}\ \delta_{ct}\ \gamma_{W_{a_g}}{}^1(t)\ \delta_r^g)$$

$$+ (\Omega_{W_a}(t)\ w_{C_a}{}^c(t)\ \beta_{G_{a_2}}{}^b(t)\ a_{G_{a_b}}{}^d(t)\ \gamma_{B_d}{}^f(t)\ e^{rst}\ \delta_{fs}\ \delta_{ct}\ \gamma_{W_{a_g}}{}^1(t)\ \delta_r^g)$$

$$+ \Omega_{W_a}(t)\ v_{W_a}(t)\ \beta_{G_{a_2}}{}^b(t)\ a_{G_{a_b}}{}^d(t)\ \gamma_{B_d}{}^f(t)\ e^{rs3}\ \delta_{fs}\ \gamma_{W_{a_g}}{}^1(t)\ \delta_r^g))$$

$$+ \bar{I}_{W_{a_2}}(t)\ ((\Omega_B{}^b(t)\ w_{C_a}{}^c(t)\ \gamma_{B_b}{}^d(t)\ e^{fgh}\ \delta_{dg}\ \delta_{ch}\ \gamma_{W_{a_m}}{}^2(t)\ \delta_f^m)$$

$$+ (\Omega_{G_a}(t)\ w_{C_a}{}^c(t)\ a_{G_{a_3}}{}^2(t)\ \gamma_{B_d}{}^f(t)\ e^{rst}\ \delta_{fs}\ \delta_{ct}\ \gamma_{W_{a_m}}{}^2(t)\ \delta_r^m))$$

$$+ \bar{I}_{W_{a_3}}(t)\ v_{W_a}'(t).$$

The a gear wheel has reached a spin up condition at the time t_{SU} if

$$(\bar{r}_{A_a}'(t_{SU}) + \bar{r}_{FA_a}'(t_{SU})) \cdot \bar{I}_{W_{a_1}}(t_{SU}) = 0.$$

For $t < t_{SU}$ the ground force parallel to the ground plane is in the direction of the vector

$$(- \bar{r}_{A_a}'(t) \cdot \bar{I}_{W_{a_1}}(t) - \bar{r}_{FA_a}'(t) \cdot \bar{I}_{W_{a_1}}(t))\ \bar{I}_{W_{a_1}}(t)$$
$$+ (- \bar{r}_{A_a}'(t) \cdot \bar{I}_{W_{a_2}}(t) - \bar{r}_{FA_a}'(t) \cdot \bar{I}_{W_{a_2}}(t))\ \bar{I}_{W_{a_2}}(t)$$

$$= - A(t)\ \bar{I}_{W_{a_1}}(t) - B(t)\ \bar{I}_{W_{a_2}}(t).$$

Now suppose that ν_a is the simple graph such if t is in $[0, t_{SU})$,

$$\nu_a(t) = \tan^{-1} \frac{B(t)}{A(t)}$$

at the time t.

Therefore, since the slip ratio at the time t is defined by

$$R_a(t) = \frac{(\bar{r}_{A_a}'(t) + \bar{r}_{FA_a}'(t)) \cdot \bar{I}_{W_{a_1}}}{\bar{r}_{A_a}'(t) \cdot \bar{I}_{W_{a_1}}},$$

and since the ground force vector may be expressed in terms of its components in the wheel plane reference system by

$$\bar{F}_{G_a}(t) = \hat{F}_{G_a}{}^{b}(t)\ \bar{I}_{W_{a_b}}(t),$$

it follows that

$$\hat{F}_{G_a}{}^{1} = -\mu_{G_a}[R_a]\ F_{G_a}{}^{3}\ \cos[\nu_a],$$

$$\hat{F}_{G_a}{}^{2} = -\mu_{G_a}[R_a]\ F_{G_a}{}^{3}\ \sin[\nu_a],$$

$$\hat{F}_{G_a}{}^{3} = F_{G_a}{}^{3}.$$

For $t > t_{SU}$ the ground force component parallel to $\bar{I}_{W_{a_1}}(t)$ is derived from the wheel and the tire rolling resistance. The ground force component parallel to $\bar{I}_{W_{a_2}}(t)$ is derived from the tire cornering effect.

If each of C and D is a simple graph such that if $t > t_{SU}$,

$$C(t) = \bar{r}_{A_a}'(t) \cdot \bar{I}_{W_{a_1}}(t),$$

and

$$D(t) = \bar{r}_{A_a}{}'(t) \cdot \bar{I}_{W_{a_2}}(t)$$

at the time t, the cornering angle of the a gear tire at a time t may be approximated by

$$\nu_{C_a}(t) = \tan^{-1} \frac{D(t)}{C(t)}.$$

It follows, therefore, that

$$\hat{F}_{G_a}{}^1 = -\mu_{R_a} |F_{G_a}{}^3|,$$

$$\hat{F}_{G_a}{}^2 = -F_{C_a}[\nu_{C_a}, F_{G_a}{}^3],$$

$$\hat{F}_{G_a}{}^3 = F_{G_a}{}^3.$$

APPENDIX F

DERIVATION OF THE ARRESTING HOOK FORCE EQUATIONS

The following derivation of the arresting hook force equations will accommodate yawed, rolled and drift landings in addition to landings where the arresting hook engages off the midpoint of the cable. In this derivation it is assumed that the arresting hook and the cable are massless. Therefore, the orientation of the arresting hook and the cable may be completely determined from the dynamics of the airframe and the geometry of the arresting hook and the cable arresting system. The arresting force from the cable is assumed to be a function of the cable runout only. It is further assumed that there is friction between the arresting hook and the cable interface. The derivation of the equations for the arresting hook forces accounts for both slipping and not slipping of the arresting hook on the cable.

Some of the equations involved are nonlinear algebraic and trigonometric relations for which a closed form solution is not viable. For these cases an iterative approach is described that may be used to obtain a solution.

The points on the arresting hook, cable and the ground that need to be identified are described as follows:

H is the point of attachment of the arresting hook to the airframe.

HP is in the arresting hook plane of symmetry and at the center of the cable

cross section subsequent to cable engagement.

HC is the point on the ground such that the straight line perpendicular to the ground containing the point HP contains the point HC.

LP is the lateral pivot point in the arresting hook shank.

SL is the point of exit of the cable from the left hand cable sheave.

SR is the point of exit of the cable from the right cable sheave. The straight line containing SL and SR is assumed to be parallel to the straight line containing the ground fixed unit vector $\bar{I}_2$ which is defined in the main body of the report.

S is the midpoint between SL and SR.

Q is the ground reference point.

N is the airframe reference point.

The numbers required for the calculation of the arresting force are defined as follows:

t_C is the time at which the arresting hook engages the cable.

d_{BAR} is the horizontal travel (parallel to $\bar{I}_1$) of the point HP from time equal to zero to time equal to t_C.

RO_M is the maximum runout of the arresting cable system.

ϕ_{H_0} is the initial (i.e. at a time t before contact with the ground or cable engagement) angle between the arresting hook shank and airframe unit vector $\bar{i}_{B_3}(t)$.

μ_C is the friction coefficient between the cable and the arresting hook.

bc is the cable semispan (half of the distance between the points SL and SR).

The following vector is required to establish the cable geometry:

$\bar{v}_S$ is the vector parallel to $\bar{I}_3$ from the point S to the ground and v_S is the minimum distance from the point S to the ground. Therefore,

$\bar{v}_S = v_S \bar{I}_3$.

The following vector functions are required in the derivation of the arresting hook force equations:

$\bar{I}_{H_1}$, $\bar{I}_{H_2}$ and $\bar{I}_{H_3}$ are orthogonal unit vector functions such that if $t \geq 0$, $\bar{I}_{H_1}(t)$ and $\bar{I}_{H_2}(t)$ are fixed in the plane of the cable with $\bar{I}_{H_2}(t) = \bar{I}_2$, $\bar{I}_{H_3}(t)$ is normal to the plane of the cable and directed down relative to the pilot and $\bar{I}_{H_1}(t) = \bar{I}_{H_2}(t) \times \bar{I}_{H_3}(t)$ at the time t.

$\bar{I}_L$ is the vector function such that if $t \geq 0$, $\bar{I}_L(t)$ is the vector from the

point HP to the point SL at the time t.

l_L is the simple graph such that if $t \geq 0$, $l_L(t)$ is the magnitude of the vector $\bar{l}_L(t)$ at the time t.

$\bar{l}_R$ is the vector function such that if $t \geq 0$, $\bar{l}_R(t)$ is the vector from the point HP to the point SR at the time t.

l_R is the simple graph such that if $t \geq 0$, $l_R(t)$ is the magnitude of the vector $\bar{l}_R(t)$ at the time t.

$\bar{RO}$ is the vector function such that if $t \geq 0$, $\bar{RO}(t)$ is the vector parallel to the unit vector $\bar{I}_{H_1}(t)$ from the straight line containing the points SL and SR to the point HP at the time t.

RO is the simple graph such that if $t \geq 0$, RO(t) is the magnitude of the vector $\bar{RO}(t)$ at the time t.

$\bar{R}_H$ is the vector function such that if $t \geq 0$, $\bar{R}_H(t)$ is the vector from the point S to the point LP at the time t.

$\bar{\xi}_H$ is the vector function such that if $t \geq 0$,

$$\bar{\xi}_H(t) = (\bar{R}_H(t) \cdot \bar{I}_{H_1}(t))\, \bar{I}_{H_1}(t).$$

ξ_H is the simple graph such that if $t \geq 0$,

$\bar{\xi}_H(t) = \xi_H(t)\, \bar{I}_{H_1}(t)$ at the time t.

$\bar{\eta}_H$ is the vector function such that if $t \geq 0$.

$\bar{\eta}_H(t) = (\bar{R}_H(t) \cdot \bar{I}_{H_2}(t))\ \bar{I}_{H_2}(t)$.

η_H is the simple graph such that if $t \geq 0$,

$\bar{\eta}_H(t) = \eta_H(t)\ \bar{I}_{H_2}(t)$ at the time t.

$\bar{r}$ is the vector function such that if $t \geq 0$, $\bar{r}(t)$ is the vector from the point Q to the jig condition location of the airframe reference point N at the time t.

d is the simple graph such that if $t \geq 0$,

$d(t) = \bar{r}(t) \cdot \bar{I}_1$ at the time t.

s is the simple graph such that if $t \geq 0$,

$s(t) = \bar{r}(t) \cdot \bar{I}_2$ at the time t.

$\bar{r}_H$ is the vector function such that if $t \geq 0$, $\bar{r}_H(t)$ is the vector from the point Q to the jig condition location of the point H at the time t.

$\bar{l}_H$ is the vector function such that if $t \geq 0$, $\bar{l}_H(t)$ is the vector from the jig condition location of the point N to the jig condition location of the point H at the time t and each of $l_H{}^b$; b = 1, 3 is a number such that

$\bar{l}_H(t) = l_H{}^b\ \bar{I}_{B_b}(t)$.

$\bar{r}_{HP}$ is the vector function such that if $t \geq 0$, $\bar{r}_{HP}(t)$ is the vector from the point Q to the point HP at the time t.

$\bar{v}_H$ is the vector function such that if $t \geq 0$,

$\bar{v}_H(t) = (\bar{r}_H(t) \cdot \bar{I}_3)\ \bar{I}_3$.

v_H is the simple graph such that if $t \geq 0$,

$\bar{v}_H(t) = v_H(t)\ \bar{I}_3$ at the time t.

$\bar{l}_{HP}$ is the vector function such that if $t \geq 0$, $\bar{l}_{HP}(t)$ is the vector, with magnitude the number l_{HP}, from the point LP to the point HP at the time t.

$\bar{l}_{LP}$ is the vector function such that if $t \geq 0$, $\bar{l}_{LP}(t)$ is the vector, with magnitude the number l_{LP}, from the jig condition location of the point H to the point LP at the time t.

$\bar{l}_{HV}$ is the vector function such that if $t \geq 0$, $\bar{l}_{HV}(t)$ is the vector, with magnitude the number l_{HV}, from the point H to the point HP at the time t.

$\bar{d}_{HP}$ is the vector function such that if $t \geq 0$,

$\bar{d}_{HP}(t) = (\bar{r}_{HP}(t) \cdot \bar{I}_1)\ \bar{I}_1$.

d_{HP} is the simple graph such that if $t \geq 0$,

$\bar{d}_{HP}(t) = d_{HP}(t)\ \bar{I}_1$ at the time t.

$\bar{v}_{HP}$ is the vector function such that if $t \geq 0$,

$\bar{v}_{HP}(t) = (\bar{r}_{HP}(t) \cdot \bar{I}_3)\ \bar{I}_3$.

v_{HP} is the simple graph such that if $t \geq 0$,

$\bar{v}_{HP}(t) = v_{HP}(t)\ \bar{I}_3$ at the time t.

$\bar{I}_{H_1}$, $\bar{I}_{H_2}$ and $\bar{I}_{H_3}$ are the unit vector functions such that if $t \geq 0$,
$\bar{I}_{H_1}(t)$ has the direction of $- \bar{I}_{HP}(t)$, $\bar{I}_{H_2}(t)$ is $\bar{I}_{B_2}(t)$, and
$\bar{I}_{H_3}(t)$ is orthogonal to $\bar{I}_{H_1}(t)$ and $\bar{I}_{H_2}(t)$ and is directed down
with respect to the pilot at the time t.

$\bar{F}_{HP}$ and $\bar{F}_H$ are the vector functions and F_{HP} and F_H are the simple graphs such that if $t \geq 0$, the force on the point HP from the arresting hook is

$$\bar{F}_{HP}(t) = (\bar{F}_{HP}(t) \cdot \bar{I}_{H_1}(t))\ \bar{I}_{H_1}(t) = - \bar{F}_H(t)$$

$$= F_{HP}(t)\ \bar{I}_{H_1}(t) = - F_H(t)\ \bar{I}_{H_1}(t) \quad \text{at the time t.}$$

$\bar{F}_{H_1}$ is the vector function and F_{H_1} and $\hat{F}_{H_1}$ are the simple graphs such that if $t \geq 0$, F_{H_1} is the component of the vector $\bar{F}_H$ such that

$$\bar{F}_{H_1}(t) = (\bar{F}_{H_1}(t) \cdot \bar{I}_{H_1}(t))\ \bar{I}_{H_1}(t)$$

$$= F_{H_1}(t)\ \bar{I}_{H_1}(t) = \hat{F}_{H_1}(RO(t))\ \bar{I}_{H_1}(t) \quad \text{at the time t.}$$

Each of $F_H{}^a$; a = 1, 3 is the simple graph such that if $t \geq 0$,

$\bar{F}_H(t) = F_H{}^a(t)\ \bar{I}_a$ at the time t.

$\bar{T}_L$ is the vector function such that if $t \geq 0$, $\bar{T}_L(t)$ is the force on the point HP from the cable segment between the points HP and SL at the time t.

$\bar{T}_R$ is the vector function such that if $t \geq 0$, $\bar{T}_R(t)$ is the force on the point HP from the cable segment between the points HP and SR at the time t.

In addition, the following simple graphs are needed in the arresting force equations:

k_{BAR} is the simple graph such that if $t \geq 0$,

$k_{BAR}(t) = |\bar{T}_L(t)| - |\bar{T}_R(t)|$ at the time t.

ϕ_H is the simple graph such that if t is in $[0,\ t_C]$, $\phi_H(t)$ is the angle between the arresting hook shank and the unit vector $\bar{I}_{B_3}(t)$ at the time t.

$\hat{F}_T$ is the simple graph such that if x is in $[0,\ RO_M]$, $\hat{F}_T(x)$ is the magnitude of the arresting force at the cable runout x.

α_L is the simple graph such that if $t \geq 0$, $\alpha_L(t)$ is the angle measured clockwise from the straight line that includes the points HP and SL to the straight line that includes the points SL and S at the time t.

α_R is the simple graph such that if $t \geq 0$, $\alpha_R(t)$ is the angle measured counterclockwise from the straight line that includes the points HP and SR

to the straight line that includes the points SR and S at the time t.

ϵ_C is the simple graph such that if $t \geq 0$, $2\epsilon_C(t)$ is the angle measured counterclockwise from the straight line that includes the points HP and SL to the straight line that includes the points HP and SR at the time t.

The cable bisector line is the straight line that is in the plane of the cable and contains the point HP and at the time t, $\epsilon_C(t)$ is the angle measured clockwise from this line to the straight line containing the points SL and HP.

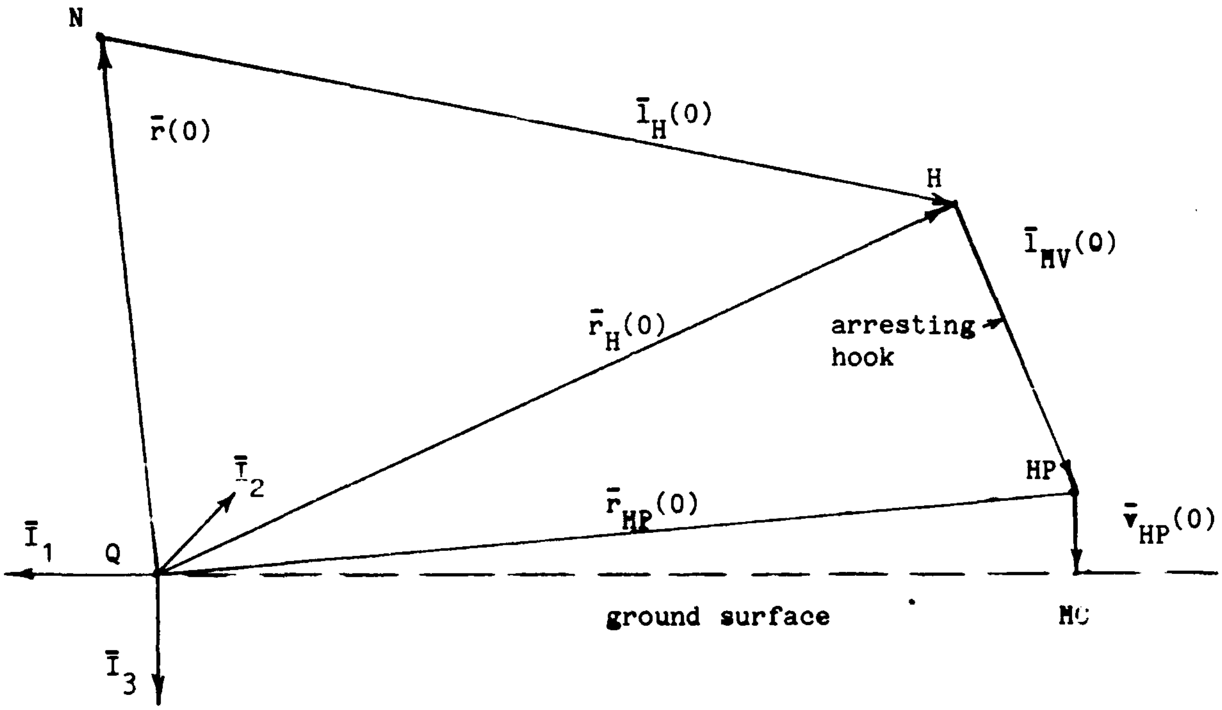
ϵ_H is the simple graph such that if $t \geq 0$, $\epsilon_H(t)$ is the angle measured clockwise from the cable bisector line to the straight line that includes the points LP and HP at the time t.

ϵ_{H_L} is the simple graph such that if $t \geq 0$, $|\epsilon_{H_L}(t)|$ is the largest number for the magnitude of the angle $\epsilon_H(t)$ for which the hook will not slip on the cable at the time t.

$\hat{\epsilon}_H$ is the simple graph such that if $t \geq 0$, $\hat{\epsilon}_H(t)$ has the magnitude $|\epsilon_{H_L}(t)|$ and the sign of $\epsilon_H(t)$ at the time t.

α_C is the simple graph such that if $t \geq 0$, $\alpha_C(t)$ is the angle, measured positive from the ground plane, that is formed by the intersection of the ground plane, the cable plane and plane which is normal to both the ground and cable planes at the time t.

If the effects of the airframe deformation are neglected, the orientation of the hook at the time $t = 0$ may be represented by the diagram below. In this diagram the point N, H and HP are coplanar.



Therefore, it is seen that if

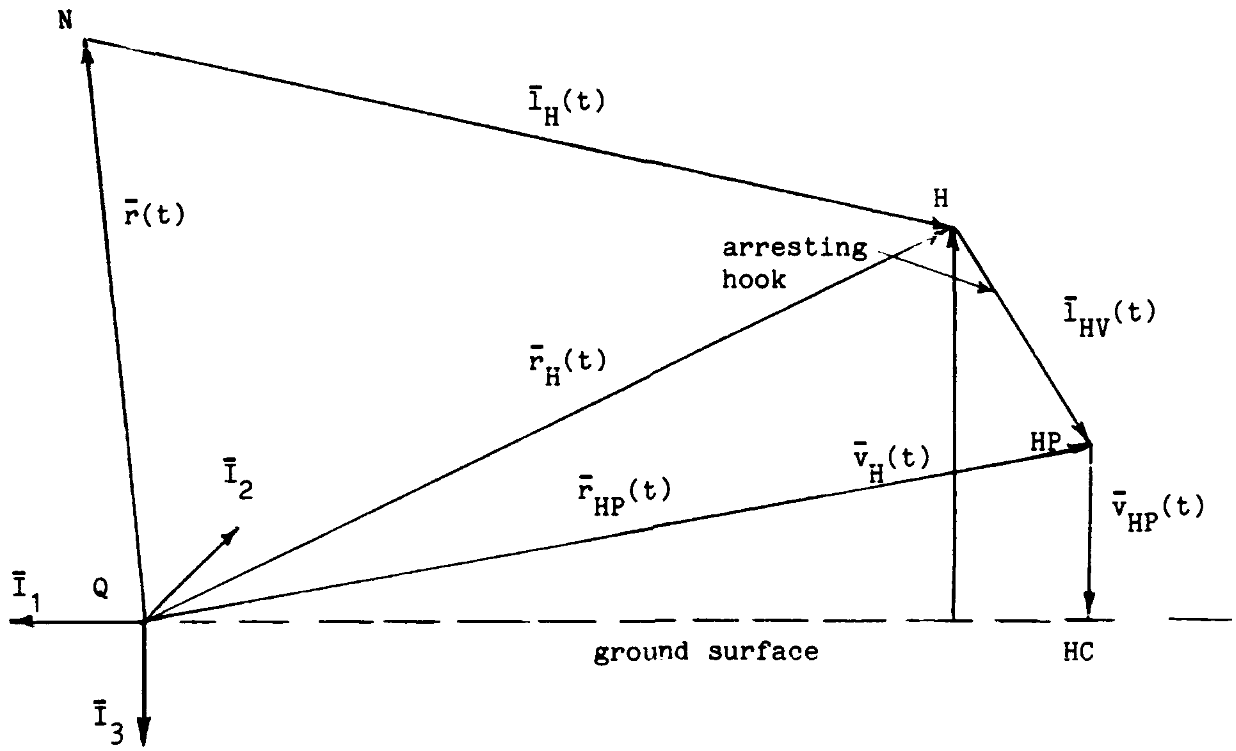
$$\bar{l}_H(0) = l_H^{\ b}\ \bar{i}_{B_b}(0),$$

then

$$d_{HP}(0) = \bar{r}_{HP}(0) \cdot \bar{I}_1 = d(0) + (l_H^{\ b} - \delta^{1b}\ l_{HV}\ \sin(\phi_{H_0})$$

$$+ \delta^{3b}\ l_{HV}\ \cos(\phi_{H_0}))\ \gamma_{B_b}^{\ 1}(0).$$

If the hook is above the ground for $t < t_C$ then the diagram below may be used to orient the hook.



From the diagram above it is seen that

$\bar{r}_{HP}(t) = \bar{r}_H(t) + \bar{l}_{HV}(t)$,

and it follows that

$d_{HP}(t) = \bar{r}_{HP}(t) \cdot \bar{I}_1 = d(t) + (l_H{}^b - \delta^{1b}\, l_{HV} \sin(\phi_{H_0})$

$+ \delta^{3b}\, l_{HV} \cos(\phi_{H_0}))\ \gamma_{B_b}{}^1(t)$.

If the hook is in contact with the ground at a time $t \leq t_C$ then the angle $\phi_H(t)$ is determined as follows: Suppose that each of a and b is a simple graph such that if t is in $[0, t_C]$, the vector $\bar{l}_{HV}(t)$ may be expressed by

$\bar{l}_{HV}(t) = a(t)\ \bar{i}_{B_1}(t) + b(t)\ \bar{i}_{B_3}(t)$ at the time t,

and if it assumed that the point HP remains a fixed distance above the ground until the cable is picked up by the arresting hook, that is

$$\bar{v}_{HP}(t) = \bar{v}_{HP}(t_C),$$

then

$$v_H(t) = \bar{r}_H(t) \cdot \bar{I}_3 = - \bar{l}_{HV}(t) \cdot \bar{I}_3 - \bar{v}_{HP}(t_C) \cdot \bar{I}_3 = v(t) + l_H{}^b \, \gamma_{B_b}{}^3(t).$$

Thus, it is evident that

$$v_H(t) = - a(t) \, \gamma_{B_1}{}^3(t) - b(t) \, \gamma_{B_3}{}^3(t) - v_{HP}(t_C),$$

and

$$|\bar{l}_{HV}(t)|^2 = a(t)^2 + b(t)^2 = (l_{HV})^2.$$

Consequently,

$$a^2(t) \left[\left(\frac{\gamma_{B_1}{}^3(t)}{\gamma_{B_3}{}^3(t)}\right)^2 + 1\right] + a(t) \left[\frac{2 \, v_H(t) \, \gamma_{B_1}{}^3(t)}{(\gamma_{B_3}{}^3(t))^2} + \frac{2 \, \gamma_{B_1}{}^3(t) \, v_{HP}(t_C)}{(\gamma_{B_3}{}^3(t))^2}\right]$$

$$+ \left[\left(\frac{v_H(t)}{\gamma_{B_3}{}^3(t)}\right)^2 - l_{HV}{}^2 + \frac{2 \, v_H(t) \, v_{HP}(t_C)}{(\gamma_{B_3}{}^3(t))^2} + \left(\frac{v_{HP}(t_C)}{\gamma_{B_3}{}^3(t)}\right)^2\right] = 0,$$

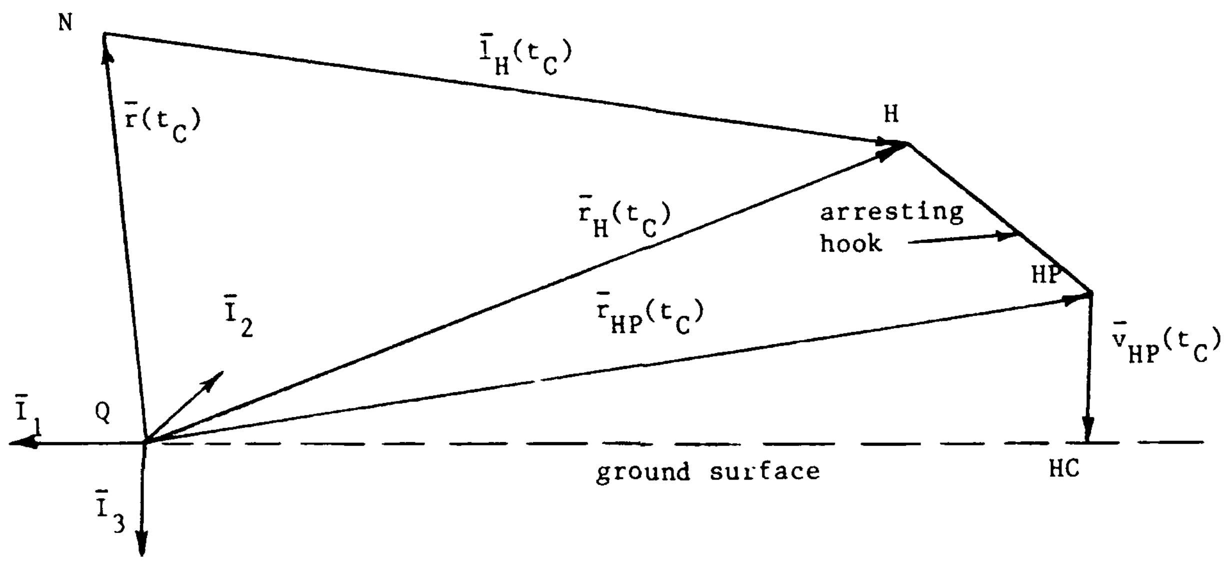
and

$$b(t) = \frac{- (v_H(t) + v_{HP}(t_C)) - a(t) \, \gamma_{B_1}{}^3(t)}{\gamma_{B_3}{}^3(t)}.$$

Therefore,

$$\tan(\phi_H(t)) = -\frac{a(t)}{b(t)}.$$

The position of the hook at the time of cable engagement with the hook (i.e. at time $t = t_C$) is shown below.



At this time the point HP is at the height $v_{HP}(t_C)$ above the ground and also

$$d_{HP}(t_C) = \bar{r}_{HP}(t_C) \cdot \bar{I}_1$$

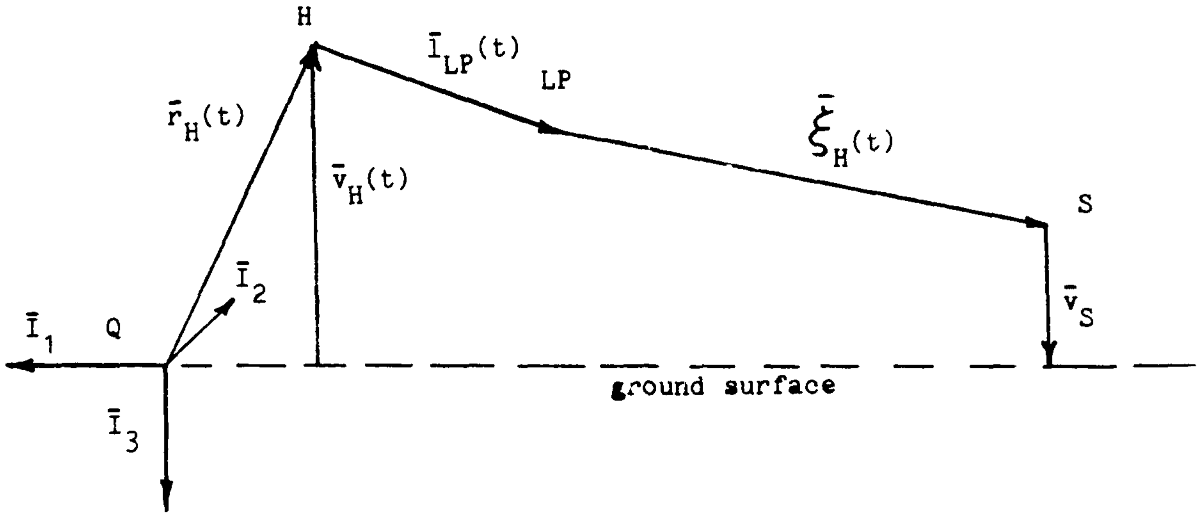
$$= d(t_C) + (l_H{}^b - \delta^{1b}\, l_{HV} \sin(\phi_H(t_C)) + \delta^{3b}\, l_{HV} \cos(\phi_H(t_C))\ \gamma_{B_b}{}^1(t_C).$$

The number d_{BAR} is defined as

$$d_{BAR} = d_{HP}(t_C) - d_{HP}(0).$$

After the cable has made engagement with the hook, it may now pivot

about the lateral pivot point LP. The simple graph ξ_H may now be determined from the geometric arrangement shown in the diagram below.



It is seen that

$$(\bar{I}_{LP}(t) + \bar{\xi}_H(t) \cdot \bar{I}_3 = - v_H(t) - v_S,$$

and

$$(\bar{I}_{LP}(t) + \bar{\xi}_H(t)) \cdot \bar{I}_1 = - (\bar{r}_H(t) \cdot \bar{I}_1 - d_{HP}(t_C)).$$

Also, it is supposed that

$$(\bar{I}_{B_2}(t) \times \bar{I}_{LP}(t)) \cdot \bar{\xi}_H(t) = 0.$$

Thus, with

$$|\bar{l}_{LP}(t)| = l_{LP},$$

$$|\bar{\xi}_H(t)| = \xi_H(t),$$

and

$$d_H(t) = \bar{r}_H(t) \cdot \bar{I}_1 = d(t) + l_H{}^b \gamma_{B_b}{}^1(t),$$

$$\bar{l}_{LP}(t) = a_1(t)\, \bar{I}_{B_1}(t) + b_1(t)\, \bar{I}_{B_3}(t),$$

$$\bar{\xi}_H(t) = a_2(t)\, \bar{I}_1(t) + b_2(t)\, \bar{I}_3,$$

it follows that

$$a_1(t)\, \gamma_{B_1}{}^3(t) + b_1(t)\, \gamma_{B_3}{}^3(t) + b_2(t) = -\, v_H(t) - v_S, \tag{F-1}$$

$$a_1(t)\, \gamma_{B_1}{}^1(t) + b_1(t)\, \gamma_{B_3}{}^1(t) + a_2(t) = -\,(d_H(t) - d_{HP}(t_C)), \tag{F-2}$$

$$a_1{}^2(t) + b_1{}^2(t) = l_{LP}{}^2, \tag{F-3}$$

$$-\, \gamma_{B_3}{}^1(t)\, a_1(t)\, a_2(t) + \gamma_{B_1}{}^1(t)\, b_1(t)\, a_2(t) \tag{F-4}$$

$$-\, \gamma_{B_3}{}^3(t)\, a_1(t)\, b_2(t) + \gamma_{B_1}{}^3(t)\, b_1(t)\, b_2(t) = 0,$$

$$\xi_H{}^2(t) = a_2{}^2(t) + b_2{}^2(t). \tag{F-5}$$

The following procedure is used for solution of these equations:

(1) Assume a number for $b_1(t)$.

(2) Calculate $a_1(t)$ from equation (F-3) (it is assumed that $a_1(t)$ is negative).

(3) Calculate $b_2(t)$ from equation (F-1).

(4) Calculate $a_2(t)$ from equation (F-2).

(5) Calculate a new candidate for $b_1(t)$ from equation (F-4).

(6) Go to step (2) of this process.

This procedure is terminated when the change in $b_1(t)$ is within the required precision.

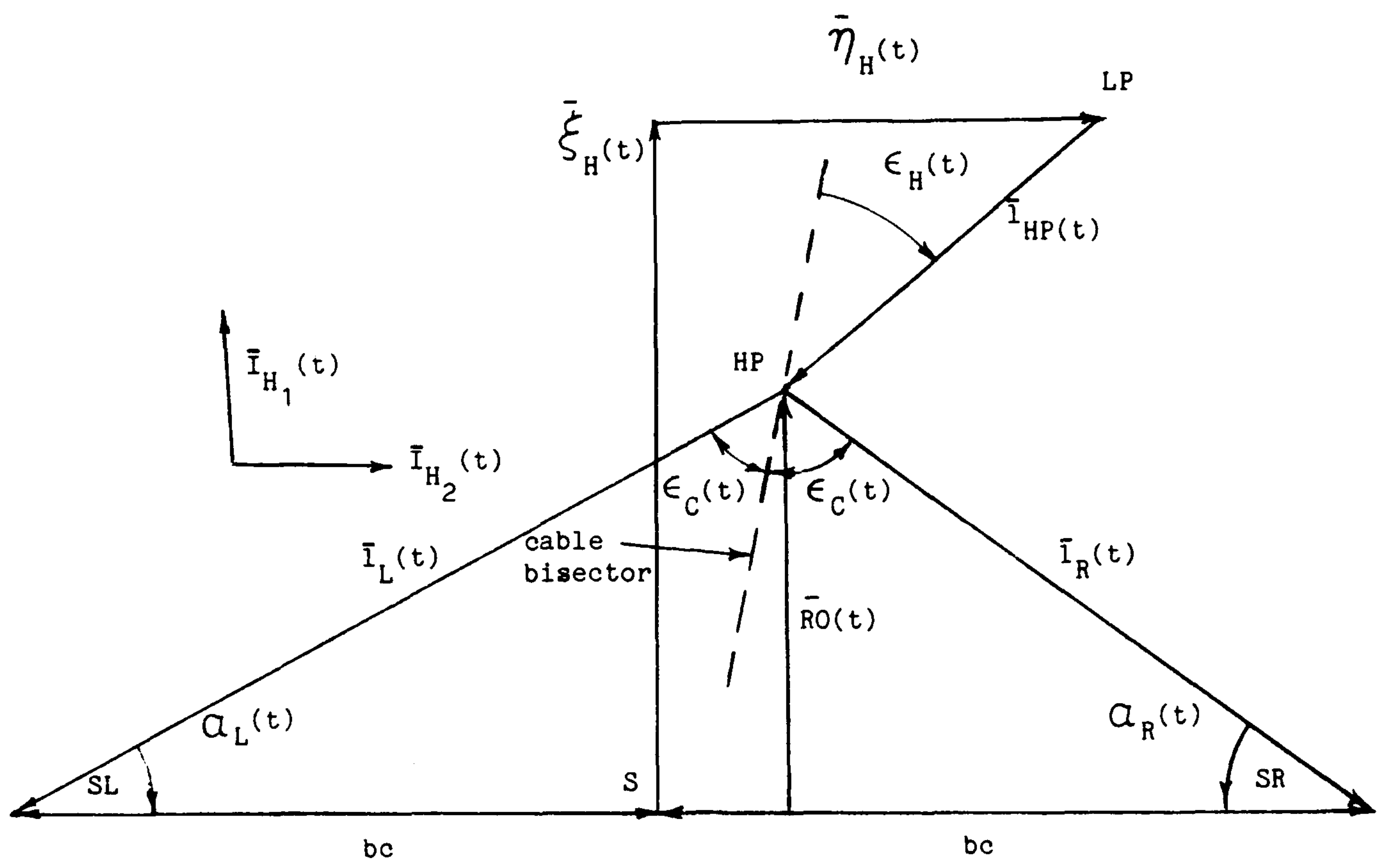
The simple graph η_H is determined from the equation

$$\eta_H(t) = (\bar{r}(t) + \bar{I}_H(t) + \bar{I}_{LP}(t)) \cdot \bar{I}_2,$$

from which the following expression is derived:

$$\eta_H(t) = s(t) + l_H^b(t)\ \gamma_{B_b}^2(t) + a_1(t)\ \gamma_{B_1}^2(t) + b_1(t)\ \gamma_{B_3}^2(t).$$

The cable geometry at a time t in the plane of the $\bar{I}_{H_1}(t)$ and $\bar{I}_{H_2}(t)$ vectors is shown in the diagram below.



Therefore, since

$$(\bar{I}_{HP}(t) + \bar{I}_L(t)) \cdot \bar{I}_{H_1}(t) = -\xi_H(t),$$

$$(\bar{I}_{HP}(t) + \bar{I}_R(t)) \cdot \bar{I}_{H_1}(t) = -\xi_H(t),$$

$$(\bar{I}_{HP}(t) + \bar{I}_L(t)) \cdot \bar{I}_{H_2}(t) = -(bc + \eta_H(t)),$$

$$(\bar{I}_{HP}(t) + \bar{I}_R(t)) \cdot \bar{I}_{H_2}(t) = (bc - \eta_H(t)),$$

it follows that after the indicated vector operations are performed,

$$l_{HP} \sin(\alpha_L(t) + \epsilon_C(t) - \epsilon_H(t)) + l_L(t) \sin(\alpha_L(t)) = \xi_H(t), \qquad \text{(F-6)}$$

$$l_{HP} \sin(\alpha_L(t) + \epsilon_C(t) - \epsilon_H(t)) + l_R(t) \sin(\alpha_R(t)) = \xi_H(t), \qquad \text{(F-7)}$$

$$l_{HP} \cos(\alpha_L(t) + \epsilon_C(t) - \epsilon_H(t)) + l_L(t) \cos(\alpha_L(t)) = bc + \eta_H(t), \quad \text{(F-8)}$$

$$- l_{HP} \cos(\alpha_L(t) + \epsilon_C(t) - \epsilon_H(t)) + l_R(t) \cos(\alpha_R(t)) = bc - \eta_H(t). \quad \text{(F-9)}$$

From equations (F-6) and (F-8) it is seen that

$$2\,(bc + \eta_H(t))\, l_L(t) \cos(\alpha_L(t)) + 2\, \xi_H(t)\, l_L(t) \sin(\alpha_L(t))$$

$$+ (l_{HP}^2 - \xi_H^2(t) - (bc + \eta_H(t))^2 - l_L^2(t)) = 0,$$

and with

$$x(t) = 2\,(bc + \eta_H(t))\, l_L(t),$$

$$y(t) = 2\, \xi_H(t)\, l_L(t),$$

$$z(t) = (l_{HP}^2 - \xi_H^2(t) - (bc + \eta_H(t))^2 - l_L^2(t)),$$

it follows that

$$x(t) \cos(\alpha_L(t)) + y(t) \sin(\alpha_L(t)) \quad z(t) = 0,$$

and consequently

$$(x^2(t) + y^2(t)) \sin^2(\alpha_L(t)) + 2\, y(t)\, z(t) \sin(\alpha_L(t)) \quad \text{(F-10)}$$

$$+ (z^2(t) - x^2(t)) = 0.$$

If equation (F-6) is substituted into equation (F-7) and equation (F-8) is substituted into equation (F-9) it is found that

$$l_L(t) \sin(\alpha_L(t)) - l_R(t) \sin(\alpha_R(t)) = 0,$$

$$l_L(t) \cos(\alpha_L(t)) + l_R(t) \cos(\alpha_R(t)) = 2\, bc.$$

Thus,

$$\tan(\alpha_R(t)) = \frac{\sin(\alpha_L(t))}{\dfrac{2\ bc}{l_{L(t)}} - \cos(\alpha_L(t))}, \tag{F-11}$$

and

$$l_R(t) = \frac{l_L(t)\ \sin(\alpha_L(t))}{\sin(\alpha_R(t))}. \tag{F-12}$$

Suppose that $|\epsilon_{H_L}(t)|$ (the limit hook angle with respect to the cable bisector at the time t) is the largest number for $|\epsilon_H(t)|$ for which the hook will not slip on the cable.

Therefore, if

$|\epsilon_H(t)|$ is less than or equal to $|\epsilon_{H_L}(t)|$,

then the hook will not slip on the cable and consequently

$$l_L(t) - l_R(t) = k_{BAR}(t). \tag{F-13}$$

The numbers $l_L(t)$, $l_R(t)$, $\alpha_L(t)$ and $\alpha_R(t)$ for this case may be obtained as follows:

(1) Assume a number for $l_L(t)$.

(2) Calculate $\alpha_L(t)$ from equation (F-10).

(3) Calculate $\alpha_R(t)$ from equation (F-11).

(4) Calculate $l_R(t)$ from equation (F-12).

(5) Calculate a new candidate for $l_L(t)$ from equation (F-13).

(6) Go to step (2) in this process.

This procedure is terminated when the change in $l_L(t)$ is within the desired precision.

The angle $\epsilon_C(t)$ may be determined from the equation

$$\epsilon_C(t) = \frac{\pi}{2} - \frac{\alpha_L(t)}{2} - \frac{\alpha_R(t)}{2}. \tag{F-14}$$

Further, the angle $\epsilon_H(t)$ may be found by combining equations (F-7) and (F-8) above as follows:

Equations (F-6) and (F-8) may be rearranged to determine

$$\sin(\alpha_L(t) + \epsilon_C(t) - \epsilon_H(t)) = \frac{\xi_H(t) - l_L(t)\sin(\alpha_L(t))}{l_{HP}},$$

and

$$\cos(\alpha_L(t) + \epsilon_C(t) - \epsilon_H(t)) = \frac{bc + \eta_H(t) - l_L(t)\cos(\alpha_L(t))}{l_{HP}}.$$

Therefore,

$$\sin(\epsilon_H(t)) = \sin(\alpha_L(t) + \epsilon_C(t))\cos(\alpha_L(t) + \epsilon_C(t) - \epsilon_H(t))$$
$$- \cos(\alpha_L(t) + \epsilon_C(t))\sin(\alpha_L(t) + \epsilon_C(t) - \epsilon_H(t)),$$

$$\cos(\epsilon_H(t)) = \cos(\alpha_L(t) + \epsilon_C(t)) \cos(\alpha_L(t) + \epsilon_C(t) - \epsilon_H(t))$$

$$+ \sin(\alpha_L(t) + \epsilon_C(t)) \sin(\alpha_L(t) + \epsilon_C(t) - \epsilon_H(t)),$$

and

$$\tan(\epsilon_H(t)) = \frac{\sin(\epsilon_H(t)}{\cos(\epsilon_H(t))}.$$

In the event that $|\epsilon_H(t)|$ as found from the equation above is greater than the limit angle $|\epsilon_{H_L}(t)|$ then the following definition is needed:

Suppose that $\hat{\epsilon}_H$ is a simple graph such that if $t \geq 0$,

$$\hat{\epsilon}_H(t) = \frac{\epsilon_H(t)}{|\epsilon_H(t)|} |\epsilon_{H_L}(t)| \text{ at the time t.}$$

With $\hat{\epsilon}_H(t)$ substituted for $\epsilon_H(t)$, equations (F-6) and (F-8) above may be combined to derive the following equation for the number $l_L(t)$:

$$l_L^2(t) = (\xi_H(t) - l_{HP} \sin(\alpha_L(t) + \epsilon_C(t) - \hat{\epsilon}_H(t)))^2 \tag{F-15}$$

$$+ (bc + \eta_H(t) - l_{HP} \cos(\alpha_L(t) + \epsilon_C(t) - \hat{\epsilon}_H(t)))^2.$$

The simple graph l_L may be determined from the following procedure:

(1) Assume a number for $l_L(t)$.

(2) Calculate $\alpha_L(t)$ from equation (F-10).

(3) Calculate $\alpha_R(t)$ from equation (F-11).

(4) Calculate $l_R(t)$ from equation (F-12).

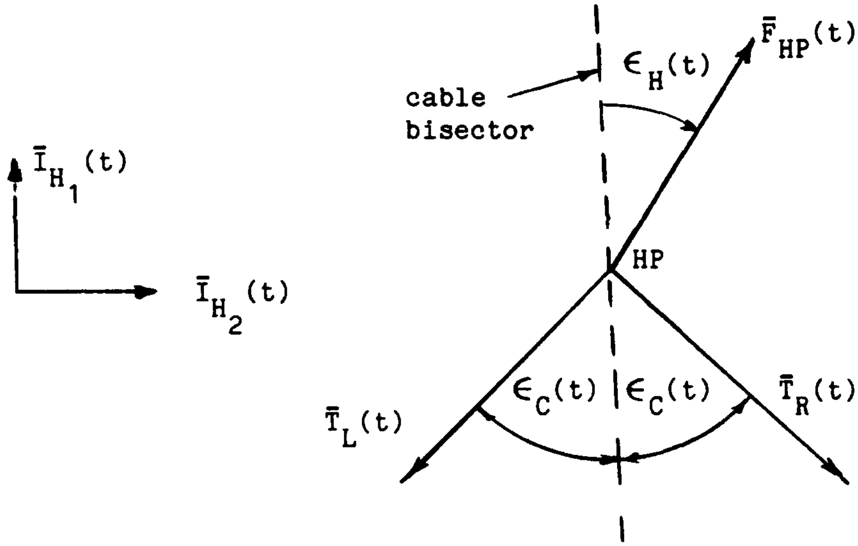
(5) Calculate $\epsilon_C(t)$ from equation (F-14).

(6) Calculate $l_L(t)$ from equation (F-15).

(7) Go to step (2) in this process.

This procedure is terminated when the change in $l_L(t)$ is within the desired precision.

The cable tensions and hook forces may be computed from an examination of the forces on the point HP as shown in the diagram below.



From this diagram it is found that

$$\bar{F}_{HP}(t) + \bar{T}_L(t) + \bar{T}_R(t) = 0,$$

or

$$|\bar{F}_{HP}(t)| \cos(\epsilon_H(t)) = |\bar{T}_R(t)| \cos(\epsilon_C(t) + |\bar{T}_L(t)| \cos(\epsilon_C(t)),$$

$$- |\bar{F}_{HP}(t)| \sin(\epsilon_H(t)) = |\bar{T}_R(t)| \sin(\epsilon_H(t)) - |\bar{T}_L(t)| \sin(\epsilon_C(t)).$$

With μ_C the friction coefficient between the hook and the cable the limiting condition for the hook to slip on the cable is determined from

$$|\bar{T}_L(t)| = |\bar{T}_R(t)| \exp(\mu_C (\pi - 2 \epsilon_C(t)))$$
$$= |\bar{T}_R(t)| \exp(\mu_C (\alpha_L(t) + \alpha_R(t)))$$

for the case when $\epsilon_H(t)$ is greater than zero and from

$$|\bar{T}_R(t)| = |\bar{T}_L(t)| \exp(\mu_C (\pi - 2 \epsilon_C(t)))$$
$$= |\bar{T}_L(t)| \exp(\mu_C (\alpha_L(t) + \alpha_R(t)))$$

for the case when $\epsilon_H(t)$ is less than zero.

It is supposed that at a time t greater than t_C there is a component of the vector $\bar{F}_H$ defined as F_{H_1} such that

$$\bar{F}_{H_1}(t) = \bar{I}_{H_1}(t) F_{H_1}(t),$$

and it is further supposed that with

$$RO(t) = l_R(t) \sin(\alpha_R(t)),$$

there is a prescribed simple graph $\hat{F}_{H_1}$ such that

$$\hat{F}_{H_1}(RO(t)) = F_{H_1}(t).$$

From this definition it follows that

$$F_{H_1}(t) = |\bar{T}_R(t)| \sin(\alpha_R(t)) + |\bar{T}_L(t)| \sin(\alpha_L(t)).$$

In the case where the cable is slipping on the hook and $\epsilon_H(t)$ is

greater than zero,

$$|\bar{T}_R(t)| = \frac{F_{H_1}(t)}{(\exp(\mu_C\ (\alpha_L(t) + \alpha_R(t))) \sin(\alpha_L(t) + \sin(\alpha_R(t))}.$$

For the case where the cable is slipping on the hook and $\epsilon_H(t)$ is less than zero,

$$|\bar{T}_L(t)| = \frac{F_{H_1}(t)}{(\exp(\mu_C\ (\alpha_L(t) + \alpha_R(t))) \sin(\alpha_R(t)) + \sin(\alpha_L(t)))}.$$

Thus, the magnitude of the axial force in the arresting hook shank is

$$|\bar{F}_{HP}(t)| = F_{HP}(t) = |\bar{T}_R(t)|^2 + |\bar{T}_L(t)|^2 + 2\ |\bar{T}_R(t)|\ |\bar{T}_L(t)| \cos(2\ \epsilon_C(t)),$$

and the number $\epsilon_{H_L}(t)$ is calculated from

$$\sin(\epsilon_{H_L}(t)) = \frac{(|\bar{T}_L(t)| - |\bar{T}_R(t)|) \sin(\epsilon_C(t))}{|\bar{F}_{HP}(t)|}.$$

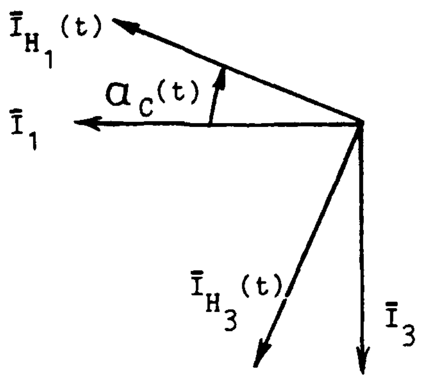
In the case where the cable is not slipping on the hook,

$$|\bar{F}_{HP}(t)| = F_{HP}(t) = \frac{F_{H_1}(t)}{\sin(\alpha_L(t) + \epsilon_C(t) - \epsilon_H(t))},$$

$$|\bar{T}_L(t)| = \frac{|\bar{F}_{HP}(t)| \sin(\epsilon_H(t) + \epsilon_C(t))}{\sin(2\ \epsilon_C(t))},$$

$$|\bar{T}_R(t)| = \frac{F_{H_1}(t) - |\bar{T}_L(t)| \sin(\alpha_L(t))}{\sin(\alpha_R(t))}.$$

The diagram below shows the relationship at the time t between the unit vectors $\bar{I}_{H_b}(t)$; b = 1, 3 and the unit vectors $\bar{I}_c$; c = 1, 3.



From this diagram it is seen that

$$\bar{I}_{H_1}(t) = \cos(\alpha_C(t))\ \bar{I}_1 - \sin(\alpha_C(t))\ \bar{I}_3,$$

$$\bar{I}_{H_2}(t) = \bar{I}_2,$$

$$\bar{I}_{H_3}(t) = \sin(\alpha_C(t))\ \bar{I}_1 + \cos(\alpha_C(t))\ \bar{I}_3,$$

where

$$\tan(\alpha_C(t)) = \frac{-\ b_2(t)}{a_2(t)}.$$

This transformation may be written in the form

$$\bar{I}_{H_a}(t) = \gamma_{H_a}{}^{b}(t)\ \bar{I}_b,$$

where

$$\gamma_{H_1}^{1}(t) = \cos(\alpha_C(t)),$$

$$\gamma_{H_1}^{3}(t) = -\sin(\alpha_C(t)),$$

$$\gamma_{H_2}^{2}(t) = 1,$$

$$\gamma_{H_3}^{1}(t) = \sin(\alpha_C(t)),$$

$$\gamma_{H_3}^{3}(t) = \cos(\alpha_C(t)),$$

and all other $\gamma_{H_b}^{a}(t) = 0.$

The transformation from the the unit vectors $\bar{I}_{H_a}(t)$; a = 1, 3 to the unit vectors $\bar{i}_{H_b}(t)$; b = 1, 3 may be expressed by

$$\bar{i}_{H_b}(t) = \alpha_{H_b}^{a}(t)\ \bar{I}_{H_a}(t).$$

From the definition of the unit vector function $\bar{i}_{H_1}$ it follows that

$$\bar{i}_{H_1}(t) = \sin(\alpha_L(t) + \epsilon_C(t) - \epsilon_H(t))\ \bar{I}_{H_1}(t)$$

$$+ \cos(\alpha_L(t) + \epsilon_C(t) - \epsilon_H(t))\ \bar{I}_{H_2}(t),$$

where

$$\alpha_{H_1}^{1}(t) = \sin(\alpha_L(t) + \epsilon_C(t) - \epsilon_H(t)),$$

$$\alpha_{H_1}^{2}(t) = \cos(\alpha_L(t) + \epsilon_C(t) - \epsilon_H(t)),$$

$$\alpha_{H_1}^{3}(t) = 0.$$

Therefore,

$$\bar{i}_{H_b}(t) = \alpha_{H_b}^{\ a}(t)\ \gamma_{H_a}^{\ c}(t)\ \bar{i}_c.$$

The unit vector function $\bar{i}_{H_3}$ may be determined from the vector equations

$$\bar{i}_{B_2}(t) \cdot \bar{i}_{H_3}(t) = 0, \tag{F-16}$$

$$\bar{i}_{LP}(t) \cdot \bar{i}_{H_3}(t) = 0,$$

or

$$(a_1(t)\ \bar{i}_{B_1}(t) + b_1(t)\ \bar{i}_{B_2}(t)) \cdot \bar{i}_{H_3}(t), \tag{F-17}$$

and

$$\bar{i}_{H_3}(t) \cdot \bar{i}_{H_3}(t) = 1. \tag{F-18}$$

After the vector operations in equations (F-16), (F-17) and (F-18) have been performed, they may be rewritten as follows:

$$\gamma_{B_2}^{\ b}(t)\ \alpha_{H_3}^{\ d}(t)\ \gamma_{H_d}^{\ c}(t)\ \delta_{bc} = 0.$$

$$(a_1(t)\ \gamma_{B_1}^{\ c}(t) + b_1(t)\ \gamma_{B_2}^{\ c}(t))\ \alpha_{H_3}^{\ d}(t)\ \gamma_{H_d}^{\ f}(t)\ \delta_{cf} = 0.$$

$$\alpha_{H_3}^{\ a}(t)\ \alpha_{H_3}^{\ b}(t)\ \delta_{ab} = 1.$$

These equations may be used to find the direction cosines $\alpha_{H_3}^{\ a}(t)$; a = 1, 3.

From the definition of the vector function $\bar{F}_H$,

$$\bar{F}_H(t) = F_H(t)\, \bar{I}_{H_1}(t) = -\, \bar{F}_{HP}.$$

In addition, $\bar{F}_H$ may be determined from

$$\bar{F}_H(t) = F_H{}^a(t)\, \bar{I}_a,$$

where

$$F_H{}^1(t) = -\, F_{HP}(t) \cos(\alpha_C(t)) \sin(\alpha_L(t) + \epsilon_C(t) - \epsilon_H(t)),$$

$$F_H{}^2(t) = -\, F_{HP}(t) \cos(\alpha_L(t) + \epsilon_C(t) - \epsilon_H(t)),$$

$$F_H{}^3(t) = F_{HP}(t) \sin(\alpha_C(t)) \sin(\alpha_L(t) + \epsilon_C(t) - \epsilon_H(t)).$$

Since these relations are true for each number t greater than t_C, it follows that

$$F_H{}^1 = -\, F_{HP} \cos[\alpha_C] \sin[\alpha_L + \epsilon_C - \epsilon_H],$$

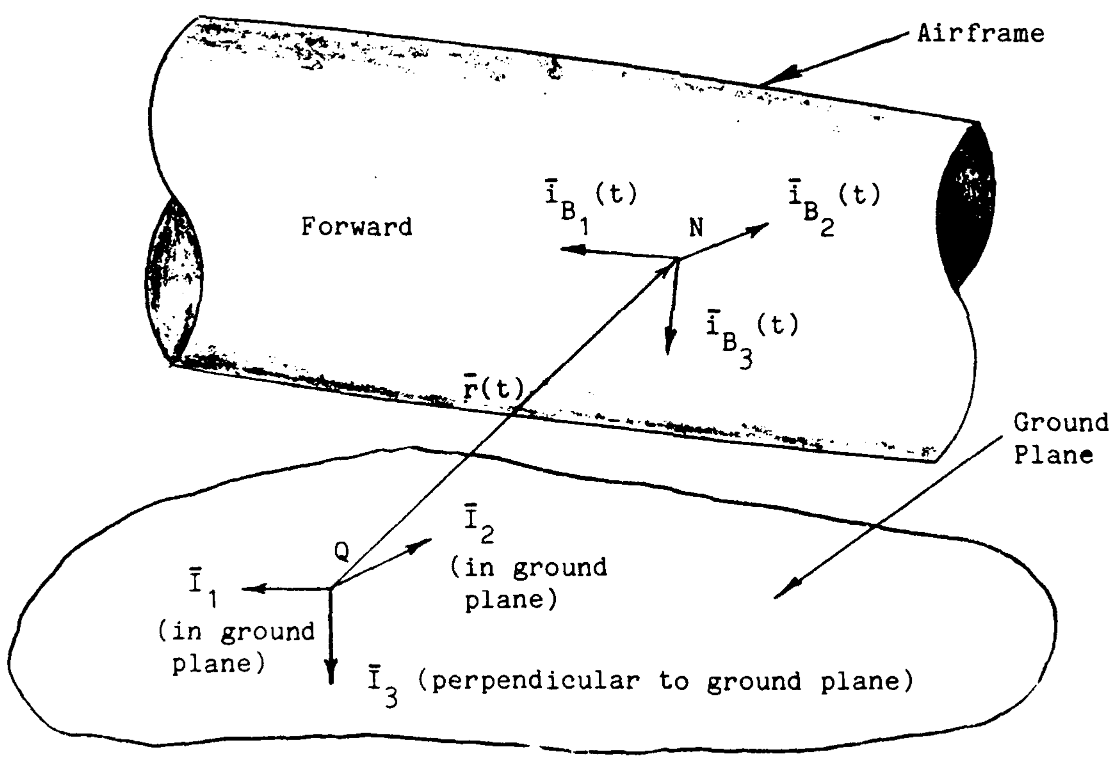
$$F_H{}^2 = -\, F_{HP} \cos[\alpha_L + \epsilon_C - \epsilon_H],$$

$$F_H{}^3 = F_{HP} \sin[\alpha_C] \sin[\alpha_L + \epsilon_C - \epsilon_H].$$

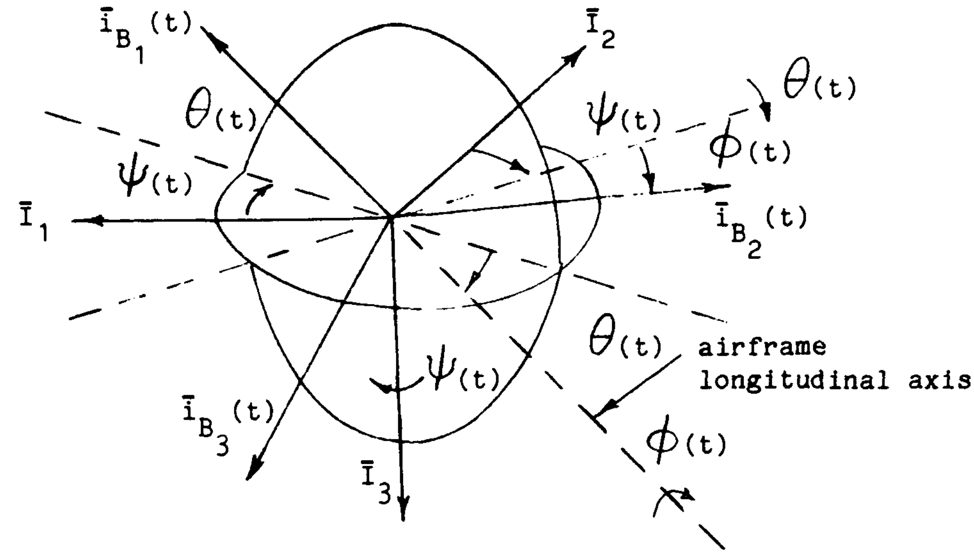
APPENDIX G

TRANSFORMATIONS

The orientation of the airframe relative to the ground is shown in the diagram below. The point Q is the ground reference point and the point N is the airframe reference point.



The transformation from the ground based unit vectors ($\bar{I}_b$; b = 1, 3) to the airframe fixed unit vectors ($\bar{i}_{B_c}$; c = 1, 3) may be established through the use of the yaw, pitch and roll Euler angles. This may be accomplished with the aid of the diagram below.



To illustrate how these Euler angles are used in the transformation from the ground based unit vectors to the airframe fixed unit vectors suppose that if $t \geq 0$, the airframe unit vectors $\bar{i}_{B_b}(t)$; b = 1, 3 are initially oriented the same as the ground based unit vectors $\bar{I}_c$; c = 1, 3. Now suppose that the unit vectors $\bar{i}_{B_1}(t)$ and $\bar{i}_{B_2}(t)$ are rotated as shown by the "yaw angle" $\psi(t)$, with the unit vector $\bar{i}_{B_3}(t)$ held fixed. In this position it is seen that

$$\bar{i}_{B_1}(t) = \cos(\psi(t))\ \bar{I}_1 + \sin(\psi(t))\ \bar{I}_2,$$

$$\bar{i}_{B_2}(t) = -\sin(\psi(t))\ \bar{I}_1 + \cos(\psi(t))\ \bar{I}_2,$$

$$\bar{i}_{B_3}(t) = \bar{I}_3.$$

From the orientation described above now suppose that the airframe unit vectors $\bar{i}_{B_1}(t)$ and $\bar{i}_{B_3}(t)$ are rotated as shown by the "pitch angle" $\theta(t)$, with the unit vector $\bar{i}_{B_2}(t)$ held fixed. In this position it is found that

$$\bar{i}_{B_1}(t) = \cos(\theta(t)) \cos(\Psi(t)) \bar{I}_1 + \cos(\theta(t)) \sin(\Psi(t)) \bar{I}_2 - \sin(\theta(t)) \bar{I}_3,$$

$$\bar{i}_{B_2}(t) = - \sin(\Psi(t)) \bar{I}_1 + \cos(\Psi(t)) \bar{I}_2,$$

$$\bar{i}_{B_3}(t) = \sin(\theta(t)) \cos(\Psi(t)) \bar{I}_1 + \sin(\theta(t)) \sin(\Psi(t)) \bar{I}_2 + \cos(\theta(t)) \bar{I}_3.$$

Finally, suppose that the airframe unit vectors $\bar{i}_{B_2}(t)$ and $\bar{i}_{B_3}(t)$ are rotated as shown by the "roll angle" $\phi(t)$, with the vector $\bar{i}_{B_1}(t)$ held fixed. In this position the airframe unit vectors are aligned with the airframe as defined in the main body of the report and the transformation from the ground based unit vectors to the airframe unit vectors is found to be

$$\bar{i}_{B_1}(t) = \cos(\theta(t)) \cos(\Psi(t)) \bar{I}_1 + \cos(\theta(t)) \sin(\Psi(t)) \bar{I}_2$$
$$- \sin(\theta(t)) \bar{I}_3,$$

$$\bar{i}_{B_2}(t) = (- \cos(\phi(t)) \sin(\Psi(t)) + \sin(\phi(t)) \sin(\theta(t)) \cos(\Psi(t))) \bar{I}_1$$
$$+ (\cos(\phi(t)) \cos(\Psi(t)) + \sin(\phi(t)) \sin(\theta(t)) \sin(\Psi(t))) \bar{I}_2$$
$$+ \sin(\phi(t)) \cos(\theta(t)) \bar{I}_3,$$

$$\bar{i}_{B_3}(t) = (\sin(\phi(t)) \sin(\Psi(t)) + \cos(\phi(t)) \sin(\theta(t)) \cos(\Psi(t))) \bar{I}_1$$

$$+ (-\sin(\phi(t))\cos(\Psi(t)) + \cos(\phi(t))\sin(\theta(t))\sin(\Psi(t)))\,\bar{I}_2$$

$$+ \cos(\phi(t))\cos(\theta(t))\,\bar{I}_3.$$

Now suppose that $\Gamma_{B_c}{}^{h}$; c = 1, 3 is a simple surface such that the transformation above may be written in the form

$$\bar{I}_{B_c}(t) = \Gamma_{B_c}{}^{h}[\theta, \phi, \Psi](t)\,\bar{I}_h, \tag{G-1}$$

or

$$\bar{I}_{B_c}(t) = \gamma_{B_c}{}^{h}(t)\,\bar{I}_h. \tag{G-2}$$

In the Kinetic Energy Formulation $\bar{r}$ was defined as the vector function such that if $t \geq 0$, $\bar{r}(t)$ is the vector from the point Q to the jig condition location of the point N at the time t. Further, $\bar{r}(t)$ was defined as

$$\bar{r}(t) = \zeta^{b}(t)\,\bar{I}_b.$$

It follows then that

$$\bar{r}'(t) = \zeta^{b\prime}(t)\,\bar{I}_b = V_B{}^{b}(t)\,\bar{I}_{B_b}(t), \tag{G-3}$$

where $V_B{}^{b}(t)$ was defined as the bth body axis component of the velocity vector.

From equation (G-3) it is seen that

$$V_B{}^{b}(t)\,\delta_{bd} = (\bar{I}_c \cdot \bar{I}_{B_d}(t))\,\zeta^{c\prime}(t).$$

But, from equation (G-2) it is found that

$$(\bar{I}_c \cdot \bar{I}_{B_d}(t)) = \gamma_{B_d}{}^{f}(t)\, \delta_{cf}.$$

Therefore,

$$v_B{}^{b}(t) = \gamma_{B_d}{}^{f}(t)\, \delta_{cf}\, \delta^{bd}\, \zeta^{c\prime}(t).$$

In the Kinetic Energy Formulation the transformation from the generalized coordinate velocities to the quasi-coordinate velocities was given as

$$v^a = \alpha^a_b\, q^{b\prime}.$$

It follows therefore for a in the interval [1, 3] and b in the interval [1, 3] that

$$\alpha^a_b = \gamma_{B_c}{}^{d}\, \delta_{bd}\, \delta^{ac}.$$

The transformation from the quasi-coordinate velocities to the generalized coordinate velocities was given by

$$q^{a\prime} = \beta^a_b\, v^b.$$

Therefore, for a in [1, 3], b in [1, 3] and c in [1, 3] it follows that

$$\beta^a_b = \gamma_{B_b}{}^{a}.$$

The α_b^a terms where a is in [4, 6] and b is in [4, 6] may be determined as follows: Since the relation in (G-1) is true for each positive number t then the vector function $\bar{i}_{B_c}$ may be differentiated to obtain

$$\bar{i}_{B_c}{}' = (\Gamma^h_{B_c;\theta}\,\theta' + \Gamma^h_{B_c;\phi}\,\phi' + \Gamma^h_{B_c;\psi}\,\psi')\,\bar{I}_h.$$

But,

$$\bar{I}_h = \gamma^f_{B_d}\,\delta_{fh}\,\delta^{dr}\,\bar{i}_{B_r},$$

and

$$\bar{i}_{B_c}{}' = \bar{\Omega}_B \times \bar{i}_{B_c}$$

$$= e^{bdf}\,\Omega_B{}^g\,\delta_{cf}\,\delta_{dg}\,\bar{i}_{B_b}.$$

Consequently,

$$e^{bdf}\,\Omega_B{}^g\,\delta_{cf}\,\delta_{dg} \tag{G-4}$$

$$= (\Gamma^h_{B_c;\theta}\,\theta' + \Gamma^h_{B_c;\phi}\,\phi' + \Gamma^h_{B_c;\psi}\,\psi')\,\gamma^f_{B_d}\,\delta_{hf}\,\delta^{bd}.$$

When the operations in equation (G-4) are performed it is found that

$$\Omega_B{}^1 = \phi' - \sin[\theta]\,\psi',$$

$$\Omega_B{}^2 = \cos[\phi]\,\theta' + \cos[\theta]\,\sin[\phi]\,\psi',$$

$$\Omega_B{}^3 = -\sin[\phi]\,\theta' + \cos[\theta]\,\cos[\phi]\,\psi'.$$

It is evident then from the definition of α^a_b, as given in the main body of the report, that the following terms may be derived:

$\alpha^4_4 = 0.$ $\alpha^4_5 = 1.$ $\alpha^4_6 = -\sin[\theta].$

$\alpha^5_4 = \cos[\phi].$ $\alpha^5_5 = 0.$ $\alpha^5_6 = \cos[\theta]\ \sin[\phi].$

$\alpha^6_4 = -\sin[\phi].$ $\alpha^6_5 = 0.$ $\alpha^6_6 = \cos[\theta]\ \cos[\phi].$

Further, for a in [4, 6], b in [4,6] and c in [4, 6], β^b_c is defined by $\alpha^a_b \beta^b_c = \delta^a_c.$

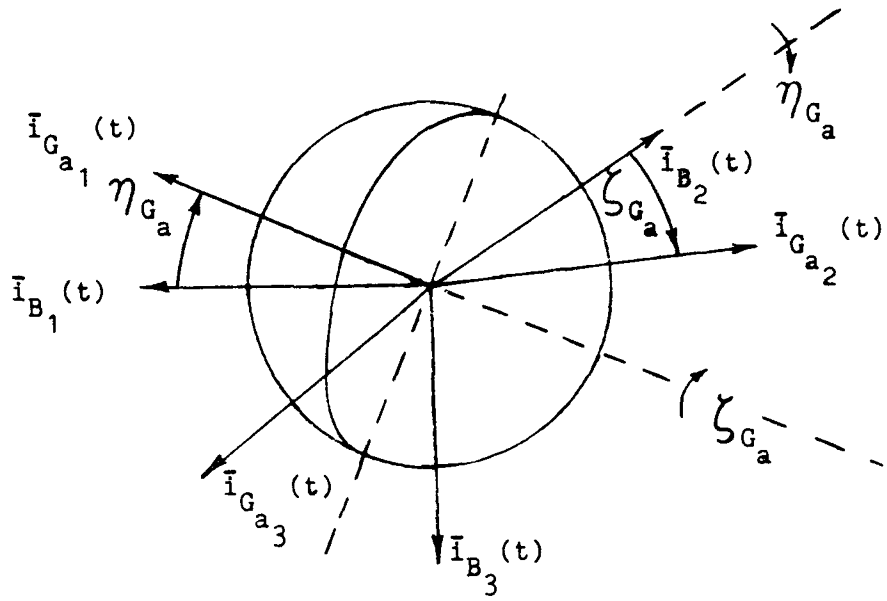
Therefore, the following relations are determined:

$\beta^4_4 = 0.$ $\beta^4_5 = \cos[\phi].$ $\beta^4_6 = -\sin[\phi].$

$\beta^5_4 = 1.$ $\beta^5_5 = \tan[\theta]\ \sin[\phi].$ $\beta^5_6 = \tan[\theta]\ \cos[\phi].$

$\beta^6_4 = 0.$ $\beta^6_5 = \sin[\phi]/\cos[\theta].$ $\beta^6_6 = \cos[\phi]/\cos[\theta].$

In many cases it is desirable to have a reference system that is fixed relative to the airframe but rotated to be aligned with a gear component such as a shock strut. Such a reference system can be defined through the use of the Euler angles η_{G_a} and ζ_{G_a} defined in the main body of the report and illustrated in the diagram below.



To establish the required transformation suppose that the at a time t the a gear unit vectors $\bar{i}_{G_{a_c}}(t)$; c = 1, 3 are aligned with the airframe fixed unit vectors $\bar{i}_{B_b}(t)$; b = 1, 3. Now suppose that the a gear fixed unit vectors $\bar{i}_{G_{a_1}}(t)$ and $\bar{i}_{G_{a_3}}(t)$ are rotated as shown below through the pitch Euler angle η_{G_a}, with $\bar{i}_{G_{a_2}}(t)$ held fixed.

In this position it is seen that

$$\bar{i}_{G_{a_1}}(t) = \cos(\eta_{G_a})\, \bar{i}_{B_1}(t) - \sin(\eta_{G_a})\, \bar{i}_{B_3}(t),$$

$$\bar{i}_{G_{a_2}}(t) = \bar{i}_{B_2}(t),$$

$$\bar{i}_{G_{a_3}}(t) = \sin(\eta_{G_a})\, \bar{i}_{B_1}(t) + \cos(\eta_{G_a})\, \bar{i}_{B_3}(t).$$

From this orientation now suppose that the a gear unit vectors $\bar{i}_{G_{a_2}}(t)$ and $\bar{i}_{G_{a_3}}(t)$ are rotated to their final position as shown in the diagram below through the roll Euler angle ζ_{G_a}, with $\bar{i}_{G_{a_1}}(t)$ held fixed.

In this alignment it is found that

$$\bar{i}_{G_{a_1}}(t) = \cos(\eta_{G_a})\, \bar{i}_{B_1} - \sin(\eta_{G_a})\, \bar{i}_{B_3}(t),$$

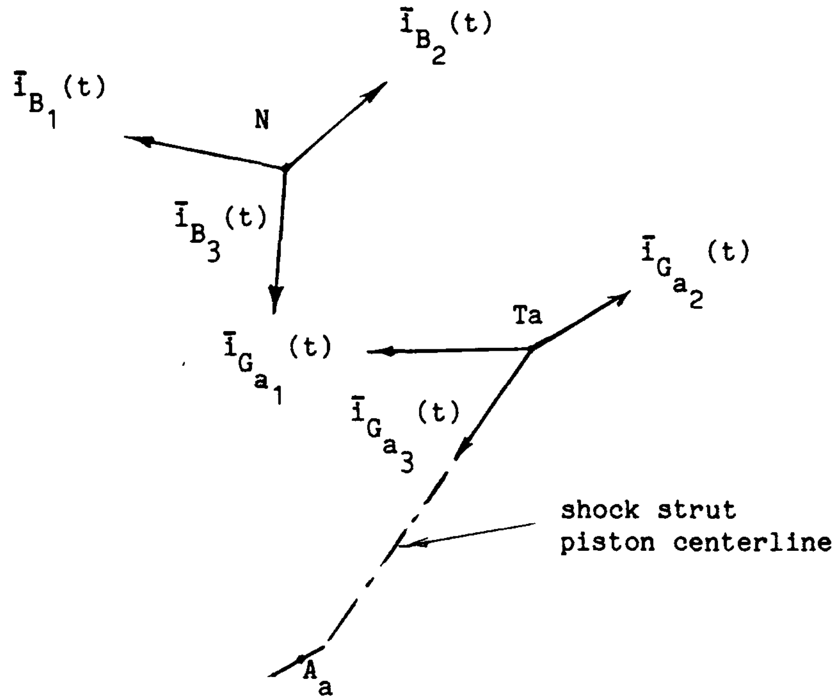
$$\bar{i}_{G_{a_2}}(t) = \sin(\eta_{G_a})\, \sin(\zeta_{G_a})\, \bar{i}_{B_1}(t) + \cos(\zeta_{G_a})\, \bar{i}_{B_2}(t)$$
$$+ \cos(\eta_{G_a})\, \sin(\zeta_{G_a})\, \bar{i}_{B_3}(t),$$

$$\bar{i}_{G_{a_3}}(t) = \sin(\eta_{G_a})\, \cos(\zeta_{G_a})\, \bar{i}_{B_1}(t) - \sin(\zeta_{G_a})\, \bar{i}_{B_2}(t)$$
$$+ \cos(\eta_{G_a})\, \cos(\zeta_{G_a})\, \bar{i}_{B_3}(t),$$

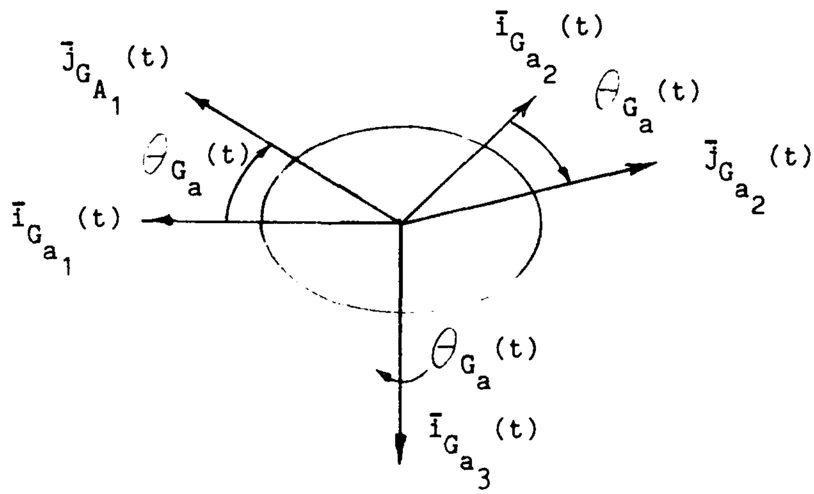
which may be written in the form

$$\bar{i}_{G_{a_c}}(t) = \gamma_{G_{a_c}}{}^{b}\, \bar{i}_{B_b}(t). \qquad \text{(G-5)}$$

The $\bar{i}_{G_{a_c}}(t)$; c = 1, 3 unit vectors are illustrated in the diagram below for the case where it is desired to have the shock strut piston centerline aligned a time t with the vector $\bar{i}_{G_{a_3}}(t)$.



For gears with a castoring degree of freedom it is convenient to have a set of unit vectors fixed in the gear. These unit vectors, $\bar{j}_{G_{a_b}}(t)$; b = 1, 3 are shown relative to the $\bar{i}_{G_{a_b}}(t)$; b = 1, 3 unit vectors in the diagram below.



It is seen that

$$\bar{i}_{G_{a_1}}(t) = \cos(\theta_{G_a}(t))\, \bar{j}_{G_{a_1}}(t) - \sin(\theta_{G_a}(t))\, \bar{j}_{G_{a_2}}(t),$$

$$\bar{i}_{G_{a_2}}(t) = \sin(\theta_{G_a}(t))\, \bar{j}_{G_{a_1}}(t) + \cos(\theta_{G_a}(t))\, \bar{j}_{G_{a_2}}(t),$$

$$\bar{i}_{G_{a_3}}(t) = \bar{j}_{G_{a_3}}(t),$$

which may be expressed as

$$\bar{i}_{G_{a_c}}(t) = \hat{\alpha}_{G_{a_c}}{}^{b}(t)\, \bar{j}_{G_{a_b}}(t). \qquad \text{(G-6)}$$

Now suppose that if $t \geq 0$, c is in [1, 3] and d is in [1, 3] that there is a transformation $\lambda_{G_{a_d}}{}^{c}(t)$ such that

$\bar{j}_{G_{a_d}}(t) = \lambda_{G_{a_d}}{}^{c}(t)\, \bar{i}_{G_{a_c}}(t)$ at the time t.

It is evident that

$$\bar{j}_{G_{a_d}}(t) \cdot \bar{i}_{G_{a_g}}(t) = \lambda_{G_{a_d}}{}^{c}(t)\, \delta_{cg} = \hat{\alpha}_{G_{a_g}}{}^{b}(t)\, \delta_{bd}.$$

Thus,

$$\lambda_{G_{a_d}}{}^{c}(t) = \hat{\alpha}_{G_{a_g}}{}^{f}(t)\, \delta_{df}\, \delta^{cg}.$$

Therefore, it is found that

$$\bar{j}_{G_{a_d}}(t) = \hat{\alpha}_{G_{a_g}}{}^{e}(t)\, \delta_{de}\, \delta^{cg}\, \bar{i}_{G_{a_c}}(t). \qquad \text{(G-7)}$$

Equations (G-5) and (G-7) may be combined to obtain the transformation

$$\bar{j}_{G_{a_d}}(t) = \hat{\alpha}_{G_{a_g}}{}^{e}\, \gamma_{G_{a_c}}{}^{b}(t)\, \delta_{de}\, \delta^{cg}\, \bar{i}_{B_b}(t).$$

Thus, if

$$\alpha_{G_{a_d}}{}^{b}(t) = \hat{\alpha}_{G_{a_g}}{}^{e}\, \gamma_{G_{a_c}}{}^{b}(t)\, \delta_{de}\, \delta^{cg},$$

then

$$\bar{j}_{G_{a_d}}(t) = \alpha_{G_{a_d}}{}^{b}(t)\, \bar{i}_{B_b}(t). \tag{G-8}$$

Equation (G-8) may be rewritten in an expanded form as

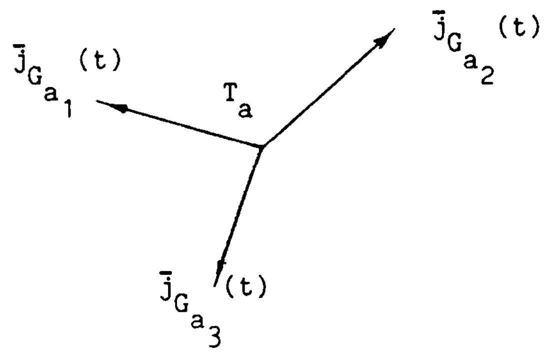
$$\bar{j}_{G_{a_1}}(t) = (\cos(\theta_{G_a}(t)) \cos(\eta_{G_a}) + \sin(\theta_{G_a}(t)) \sin(\eta_{G_a}) \sin(\zeta_{G_a}))\, \bar{i}_{B_1}(t)$$
$$+ \sin(\theta_{G_a}(t)) \cos(\zeta_{G_a})\, \bar{i}_{B_2}(t)$$
$$+ (- \cos(\theta_{G_a}(t)) \sin(\eta_{G_a}) + \sin(\theta_{G_a}(t)) \cos(\eta_{G_a}) \sin(\zeta_{G_a}))\, \bar{i}_{B_3}(t),$$

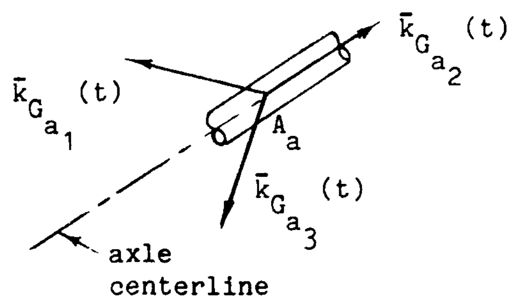
$$\bar{j}_{G_{a_2}}(t) = (- \sin(\theta_{G_a}(t)) \cos(\eta_{G_a}) + \cos(\theta_{G_a}(t)) \sin(\eta_{G_a}) \sin(\zeta_{G_a}))\, \bar{i}_{B_1}(t)$$
$$+ \cos(\theta_{G_a}(t)) \cos(\zeta_{G_a})\, \bar{i}_{B_2}(t)$$
$$+ (\sin(\theta_{G_a}(t)) \sin(\eta_{G_a}) + \cos(\theta_{G_a}(t)) \cos(\eta_{G_a}) \sin(\zeta_{G_a}))\, \bar{i}_{B_3}(t),$$

$$\bar{j}_{G_{a_3}}(t) = \sin(\eta_{G_a}) \cos(\zeta_{G_a})\, \bar{i}_{B_1}(t) - \sin(\zeta_{G_a})\, \bar{i}_{B_2}(t)$$
$$+ \cos(\eta_{G_a}) \cos(\zeta_{G_a})\, \bar{i}_{B_3}(t).$$

The a gear wheel fixed unit vectors are oriented as shown in the

diagram below





At a time t the vector $\bar{k}_{G_{a_2}}(t)$ is aligned with the axle centerline and is expressed in terms of the a gear fixed unit vectors $\bar{j}_{G_{a_b}}(t)$; b = 1, 3 by the relation

$$\bar{k}_{G_{a_2}}(t) = \beta_{G_{a_2}}{}^{b}(t)\ \bar{j}_{G_{a_b}}(t), \tag{G-9}$$

where the $\beta_{G_{a_2}}(t)$; b = 1, 3 are direction cosines determined from the rotation of the axle that is due to stroking of the a gear shock strut. The general relationship between these two sets of unit vectors is

$$\bar{k}_{G_{a_c}}(t) = \beta_{G_{a_c}}{}^{b}(t)\ \bar{j}_{G_{a_b}}(t),$$

where the unit vectors $\bar{k}_{G_{a_1}}(t)$ and $\bar{k}_{G_{a_3}}(t)$ are orthogonal to each other and

to $\bar{k}_{G_{a_2}}(t)$, but otherwise arbitrary. For the case where the a gear castors, it is usual for $\bar{k}_{G_{a_2}}(t)$ to be orthogonal to $\bar{j}_{G_{a_3}}(t)$.

The wheel-ground reference system unit vector functions $\bar{I}_{W_{a_b}}$; b = 1, 3 are defined in Appendix E in terms of the vector functions $\bar{k}_{G_{a_2}}$ and $\bar{I}_b$; b = 1, 3.

At a time t the transformations $\gamma_{H_c}{}^b(t)$ and $\alpha_{H_d}{}^c(t)$, which are defined in Appendix F, are used to relate the unit vectors $\bar{I}_b$; b = 1, 3, $\bar{I}_{H_c}(t)$; c = 1, 3 and $\bar{i}_{H_d}(t)$; d = 1, 3 through the relations

$$\bar{I}_{H_c}(t) = \gamma_{H_c}{}^b(t)\ \bar{I}_b,$$

$$\bar{i}_{H_d}(t) = \alpha_{H_d}{}^c(t)\ \bar{I}_{H_c}(t),$$

$$\bar{i}_{H_d}(t) = \alpha_{H_d}{}^e(t)\ \gamma_{H_e}{}^b(t)\ \bar{I}_b.$$